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From symplectic eigenvalues of positive definite matrices to their pseudo-orthogonal eigenvalues

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Williamson's theorem states that every real symmetric positive definite matrix A of even order can be brought to diagonal form via a symplectic T-congruence transformation. The diagonal entries of the resulting diagonal form are called the symplectic eigenvalues of A. We point at an analog of this classical result related to Hermitian positive definite matrices, *-congruences, and another class of transformation matrices, namely, pseudo-unitary matrices. This leads to the concept of pseudo-unitary (or pseudo-orthogonal, in the real case) eigenvalues of positive definite matrices.

Keywords: congruence transformation; symplectic matrix; pseudo-unitary matrix; indices of inertia; Schur inequality.

1. Introduction

Transformation of a square matrix A of the type

$$A \to Q^T AQ,$$

where Q is a nonsingular matrix, is called a T-congruence transformation or just a T-congruence. If A is complex, then another type of congruence transformations, called *-congruences, is possible:

$$A \to Q^* AQ.$$

Here, Q is again an arbitrary nonsingular matrix. If the transformation matrix Q is chosen in a special matrix class, then the term *congruence* is accordingly specified. For instance, if Q is an orthogonal (unitary) matrix, then one speaks of an orthogonal (unitary) congruence.

Let A be a real symmetric matrix. It is well known that A can be brought to diagonal form via an orthogonal congruence, which, at the same time, is a similarity:

$$Q^T AQ = D.$$

The diagonal entries d_{ii} are eigenvalues of A.

If the order of A is an even integer, $n = 2m$, then the reduction of A to diagonal form is possible via transformations from another class, namely, from the class of symplectic congruences. Recall that a matrix S of order $n = 2m$ is said to be symplectic if

$$S^T JS = J,$$

where

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

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The assertion called Williamson's theorem (see [1]) states that every real symmetric positive definite matrix of order $n = 2m$ can be brought to diagonal form via a symplectic congruence:

$$S^T AS = \Lambda. \tag{1}$$

The diagonal matrix Λ is the direct sum

$$\Lambda = D_m \oplus D_m,$$

and the diagonal entries of the $m \times m$ matrix D_m are called the symplectic eigenvalues of A. In general, they are in no way related to the conventional eigenvalues of this matrix; the only common property of the numbers in the two sets is positivity.

Our goal in this paper is to indicate that there are other groups of congruence transformations that also can be used for diagonalizing positive definite matrices.

2. Main result

We first recall a well known theorem on pairs of Hermitian matrices (see [2, Theorem 7.6.4]).

Theorem 1 *Let A and B be Hermitian $n \times n$ matrices, and let A be positive definite. Then there is a nonsingular matrix P such that $P^*AP = I_n$ and $P^*BP = M$, where M is a real diagonal matrix. The matrices B and M have the same inertias, that is, the same numbers of positive and negative eigenvalues. The diagonal entries of M are the eigenvalues of the matrix $A^{-1}B$.*

We apply this theorem to the case where

$$B = I_p \oplus (-I_q) \equiv I_{p,q}, \qquad p, q > 0, \quad p + q = n.$$

The matrix P in the relations

$$P^*AP = I_n, \qquad P^*BP = M \tag{2}$$

can be chosen so that the first p diagonal entries of M are positive.

We write M in the form

$$M = \operatorname{diag}(d_1, \dots, d_p, d_{p+1}, \dots, d_n)$$

and define the matrix

$$W = \operatorname{diag}\left(\sqrt{d_1}, \dots, \sqrt{d_p}, \sqrt{-d_{p+1}}, \dots, \sqrt{-d_n}\right).$$

Now, we perform the simultaneous congruence transformation

$$W^{-*} I_n W^{-1} = W^{-2}, \qquad W^{-*} M W^{-1} = I_{p,q} = B.$$

Combining the relations (2) and $W^{-*}MW^{-1} = B$, we obtain

$$(W^{-*}P^*)B(PW^{-1}) = B.$$

Setting

$$R = PW^{-1},$$

we have

$$R^* I_{p,q} R = I_{p,q}. \tag{3}$$

Every matrix R satisfying relation (3) is said to be pseudo-unitary. If R is real, then it is said to be pseudo-orthogonal.

We can summarize the above considerations in the following theorem.

Theorem 2 *Every Hermitian positive definite matrix A can be brought to diagonal form via a pseudo-unitary *-congruence. If A is real, then it can be brought to diagonal form via a pseudo-orthogonal congruence. In both cases, the diagonal entries of the resulting diagonal form F are the moduli of eigenvalues of the matrix $I_{p,q}A$.*

In what follows, we call the diagonal entries of F the $\{p, q\}$-pseudo-unitary (or pseudo-orthogonal) eigenvalues of the positive definite matrix A.

3. Pseudo-orthogonal versus symplectic

There is an interesting relationship between the conventional and symplectic eigenvalues of A. We first recall the Schur inequality for the eigenvalues $\lambda_i(G)$ of a complex $n \times n$ matrix G and the Frobenius norm of this matrix:

$$\sum_{i=1}^{n} |\lambda_i(G)|^2 \leq \|G\|_F^2. \tag{4}$$

The equality in (4) is attained if and only if G is a normal matrix.

Denote by σ the value

$$\sigma = \sum_{i=1}^{n} |\lambda_i(A)|^2.$$

If A is a real symmetric matrix, then

$$\sigma = \|A\|_F^2.$$

Let $d_1, \ldots, d_m$ be the symplectic eigenvalues of A. These numbers are the moduli of eigenvalues of JA, which follows from (1). Consequently, the Schur inequality yields the estimate

$$\sigma_s \stackrel{\text{def}}{\equiv} 2\sum_{i=1}^{m} d_i^2 \leq \|JA\|_F^2. \tag{5}$$

Since

$$\|JA\|_F = \|A\|_F,$$

we ultimately obtain the inequality

$$\sigma_s \leq \sigma.$$

Denote by $e_1, \ldots, e_n$ the pseudo-orthogonal eigenvalues of the positive definite matrix A. By analogy with (5), define the value

$$\sigma_{p,q} \stackrel{\text{def}}{\equiv} \sum_{i=1}^{n} e_i^2.$$

By replacing J in the above calculations by the matrix $I_{p,q}$, we get the inequality

$$\sigma_{p,q} \leq \sigma.$$

4. Numerical results

In this section, we find the collective characteristics σ, σ_s, and $\sigma_{p,q}$ of the conventional, symplectic, and pseudo-orthogonal eigenvalues for four types of symmetric test matrices. We restrict ourselves to the matrix order $n = 10$. However, since the matrix of the fourth type (the Hilbert matrix) has a cluster of very small eigenvalues, we choose its order even smaller ($n = 6$).

Type 1. All the diagonal entries of A are equal to a positive number a, while all the off-diagonal entries are equal to a real number b. The eigenvalues of such a matrix A are as follows: 1) the simple eigenvalue $a + (n-1)b$; 2) the eigenvalue $a - b$ of multiplicity $n - 1$. Thus, A is certainly positive definite if $a > b > 0$.

We set $a = 3, b = 1$. Then $\sigma \approx 180.000$ and $\sigma_s \approx 80.000$. Table 1 shows $\sigma_{p,q}$ as a function of the positive inertia index p.

Table 1

p	9	8	7	6	5	4	3	2	1
$\sigma_{p,q}$	144	116	96	84	80	84	96	116	144

It comes into notice that $\sigma_{p,q}$ is symmetric with respect to the central value $p = 5$. This is easily explained by the features of matrices of this type; namely, they are not changed by any symmetric permutation of rows and columns. As a consequence, the matrix $I_{q,p}A$ can be obtained from $-I_{p,q}A$ by a symmetric permutation of rows and columns. Neither such a permutation nor the sign reversion can change the moduli of eigenvalues.

Type 2. For all i and j, set $a_{ij} = \min(i, j)$. It is well known that such a matrix A is positive definite for all n. For $n = 10$, we have $\sigma = 2035$ and $\sigma_s = 410$. Table 2 exhibits $\sigma_{p,q}$ as a function of p.

The function $\sigma_{p,q}$ has an evident minimum at $p = 7$. The entries in the last three rows of A are the largest ones in this matrix. Here, they reverse their signs, which results in the emergence of three negative eigenvalues of the matrix $I_{7,3}A$. As the negative inertia index q increases, $\sigma_{p,q}$ grows monotonically. The number of negative eigenvalues of the matrix $I_{p,q}A$ increases along with

Table 2

p	9	8	7	6	5	4	3	2	1
$\sigma_{p,q}$	895	403	355	579	935	1315	1643	1875	1999

q. When $q = 9$, the matrix $I_{1,9}A$ can be regarded as a rank-one perturbation of $-A$; moreover, it is the entries of the upper row that are perturbed, and they have minimal sizes in the matrix. It's no wonder that $\sigma_{1,9}$ is the closest value to σ.

Type 3. A is the tridiagonal Toeplitz matrix with the diagonal entry $a \geq 2$ and the number -1 on the two neighboring diagonals. The eigenvalues of this matrix are well known and positive if $a = 2$. If $a > 2$, then the spectrum shifts to the right by $a - 2$.

For $n = 10$, we have $\sigma = 58$ and $\sigma_s = 56$. Table 3 demonstrates $\sigma_{p,q}$ as a function of p.

Table 3

p	9	8	7	6	5	4	3	2	1
$\sigma_{p,q}$	≈ 54	≈ 54	≈ 54	≈ 54	≈ 54	≈ 54	≈ 54	≈ 54	≈ 54

The fact that $\sigma_{p,q} = \sigma_{q,p}$ can be explained, similarly to matrices of type 1, by the persymmetry of A, that is, by its symmetry with respect to the antidiagonal. It is however surprising that $\sigma_{p,q}$ remains constant (within the accuracy of calculations) for all p, although the spectra of $I_{p_1,q_1}A$ and $I_{p_2,q_2}A$ are entirely different when $p_1 \neq q_2$. For now, we have no explanation to this permanence.

Type 4. For the Hilbert matrix of order $n = 6$, we have $\sigma \approx 2.68$ and $\sigma_s \approx 0.33$. Table 4 shows $\sigma_{p,q}$ as a function of p. Here, $\sigma_{p,q}$ monotonically decreases along with p.

Table 4

p	5	4	3	2	1
$\sigma_{p,q}$	≈ 2.34	≈ 1.47	≈ 1.19	≈ 0.79	≈ 0.12

5. Conclusion

In this paper, we have introduced the new notion of the pseudo-orthogonal eigenvalues of a positive definite matrix. Certain numerical results were given. Especially interesting are the results obtained for the matrix of the third type. The invariance of the Frobenius measure σ with respect to the parameters p and q remains an unexplained phenomenon. We hope to find an explanation in our future work.

References

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