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On rank decomposition and semi-symmetric rank decomposition of semi-symmetric tensors

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A tensor is called semi-symmetric if all modes but one, are symmetric. In this paper, we study the CP decomposition of semi-symmetric tensors or higher-order individual difference scaling (INDSCAL). Comon's conjecture states that for any symmetric tensor, the CP rank and symmetric CP rank are equal, while it is known that Comon's conjecture is not true in the general case but it is proved under several assumptions in the literature. In the paper, Comon's conjecture is extended for semi-symmetric CP decomposition and CP decomposition of semi-symmetric tensors under suitable assumptions. Specially, we show that if a semi-symmetric tensor has a CP rank smaller or equal to its order, or when the semi-symmetric CP rank is less than/or equal to the dimension, then the semi-symmetric CP rank is equal to the CP rank. Copyright © 2022 Shahid Beheshti University.

Keywords: INDSCAL; semi-symmetric tensor; CP decomposition; CP rank; semi-symmetric decomposition.

1. Introduction

While tensors have been in use from the early 20th century [16], the interest in tensors in the recent decade has increased significantly and tensors are applied in many different fields [21, 20, 5, 15, 2, 3]. This increasing interest is partly due to the fact that there exists a rising demand for modeling and solving more complicated problems and the fact that the computing power of computers has risen significantly in the recent decade.

Tensors have been used in various problem models and disciplines. Tensor decompositions are utilized in extracting the useful information on model entities interactions. For example, in DNA microarray data modeling using a third-order tensor, tensor decompositions can provide useful information about the genes [19].

Different types of tensor decompositions have been defined and used in literature [16]. Here, our focus is on CP decomposition and higher-order INDSCAL. Since their introduction, tensor decompositions have been used in psychometrics, chemometrics, signal processing, image processing and solving nonlinear systems of equations [21, 16, 22, 4].

It is known that for symmetric matrices, the rank and symmetric rank coincide, but this is not generally true for tensors. In 2008, Comon et. al conjectured that for any symmetric tensor, the CP rank and symmetric CP rank are equal [9], then they proceeded to prove this conjecture for special cases such as when the CP rank is less than dimension or when the CP symmetric rank is less than 3. This conjecture was later called "Comon's Conjecture" [28] and it was proved under different assumptions, afterwards Shitov proved that Comon's conjecture cannot be true in the general case [24]. Friedland proved Comon's conjecture for special cases using the rank of matricized forms [13] and Seigal proved it for cubic surfaces [23]. Also, in [28], Comon's conjecture was proved for the case of rank of the symmetric tensor is not greater than its order. Casarotti et al. investigated the additional aspects of Comon's conjecture [7].

It can be proved that for any tensor $\mathcal{A}$, it has a close relationship with a semi-symmetric tensor (for example both have the same set of eigenvalues). This corresponding semi-symmetric tensor can be used for developing methods for calculating eigenvalues

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[18]. Also, third-order semi-symmetric tensors and their CP decomposition (INDSCAL) have applications in psychometric, social sciences and signal processing [27, 8].

In this paper, some results which previously were given for symmetric CP decomposition [9, 28] will be extended for the case of the semi-symmetric CP decomposition. This means that the semi-symmetric version of Comon's conjecture is proved to be true under certain assumptions similar to those used in [9, 28] for symmetric case. We will prove that the CP rank and semi-symmetric CP rank are identical when the CP rank of a tensor is smaller or equal to the order of tensor (Theorem 1 and Theorem 2) or when the semi-symmetric CP rank is equal or less than the dimension of the tensor (Theorem 3). At the end of paper, we also extend the results for non-cubic semi-symmetric tensors.

The rest of the paper is organized as follows. Section 2 contains preliminaries on the semi-symmetric tensors, rank decomposition of tensors and other basic information. In Section 3, under assumption that the CP rank of a semi-symmetric tensor is not greater than its order or when the semi-symmetric CP rank is not greater than the dimension, we prove that the CP semi-symmetric rank is equal to the CP rank. Lastly, in Section 4, conclusions and future work are presented.

2. Preliminaries

In this paper, we denote tensors by calligraphic letters, the order of a tensor is denoted by m and n denotes the dimension of a cubic tensor. For more information about the notations, see [16]. For two non-zero vectors a and b, $a \sim b$ if and only if $a = \mu b$ for a scalar μ, otherwise $a \nsim b$. A tensor in $F^{J\times J\times \ldots \times J}$ (F can be the set of complex or real numbers) is called semi-symmetric if its elements are invariant under any permutation of all indices but the first one. More formally, $\mathcal{A} \in F^{J\times J\times \ldots \times J}$ is a semi-symmetric tensor if and only if $\mathcal{A}(i, :, \ldots, :)$ is symmetric for any $i \in \{1, \ldots, J\}$. The set of mth-order n-dimensional semi-symmetric tensors is denoted by $S_1^m(F^n)$.

Definition 1 *If $\mathcal{A} \in F^{I_1\times I_2\times \ldots \times I_m}$ is an mth-order tensor and $B \in F^{J\times I_n}$ $(n \leq m)$ is a matrix, then $\mathcal{A} \times_n B$ denotes mode-n product of $\mathcal{A}$ with B. This product is of size $I_1 \times \ldots \times I_{n-1} \times J \times I_{n+1} \times \ldots \times I_m$ and each element of it is defined as follows*

$$(\mathcal{A} \times_n B)_{i_1\ldots i_{n-1} j i_{n+1}\ldots i_m} = \sum_{i_n=1}^{I_n} a(i_1, i_2 \ldots, i_m) b(j, i_n).$$

We next present some properties of these products which will be used in developing our theory. Some of the proofs and further details that are not included here can be found in [16, 11].

Proposition 1 *Suppose $\mathcal{X} \in F^{I_1\times I_2\times \ldots \times I_m}$ is an mth-order tensor and $A \in F^{J\times I_i}$, $y \in F^{I_i}$, $z \in F^{I_j}$ then*

1) If $B \in F^{K\times J}$, then

$$\mathcal{X} \times_n A \times_n B = \mathcal{X} \times_n (BA).$$

2) If $B \in F^{I_j\times J}$ and $i \neq j$, then

$$\mathcal{X} \times_i A \times_j B = \mathcal{X} \times_j B \times_i A.$$

Definition 2 *Let $\mathcal{A} = [a(i_1, i_2, \ldots, i_m)]$ where $a(i_1, i_2, \ldots, i_m) \in F$, $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, n\}$ be an mth-order tensor and $x = (x_1, x_2, \ldots x_n)^t$ be a vector in F^n, then for $1 \leq r \leq m-1$, the tensor multiplication by a vector is defined as follows:*

$$\left(\mathcal{A}x^{m-r}\right)_{i_1\ldots i_r} := \sum_{i_{r+1},\ldots,i_m=1}^{n} a(i_1, i_2, \ldots, i_m) x_{i_{r+1}} x_{i_{r+2}} \ldots x_{i_m}.$$

Note that by the above definition, $\mathcal{A}x^m$, $\mathcal{A}x^{m-1}$ and $\mathcal{A}x^{m-2}$ are a scalar, vector and matrix, respectively.

For every tensor $\mathcal{A}$ there exists a corresponding semi-symmetric tensor $\mathcal{B}$ such that for every vector x,

$$\mathcal{A}x^{m-1} = \mathcal{B}x^{m-1}.$$

Tensor $\mathcal{B}$ inherits many properties and has a close relationship with $\mathcal{A}$ ($\mathcal{A}$ is called the mother tensor). See [18, p. 630] for the definition and more details.

Definition 3 *The outer product of vectors $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ is denoted by $a^{(1)} \circ a^{(2)} \circ \ldots \circ a^{(m)}$ and defined as follows*

$$(a^{(1)} \circ a^{(2)} \circ \ldots \circ a^{(m)})_{i_1 i_2\ldots i_m} = a^{(1)}_{i_1} \times a^{(2)}_{i_2} \times \ldots \times a^{(m)}_{i_m}.$$

Definition 4 *Let $\mathcal{A} \in F^{I_1 \times I_2 \times \ldots \times I_m}$ be an mth-order tensor. A CP decomposition is defined as*

$$\mathcal{A} = [[A^{(1)}, A^{(2)}, ..., A^{(m)}]] \equiv \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(m)} \quad (1)$$

in which, $A^{(i)}, 1 \leq i \leq m$ is an $I_i \times R$ matrix and $a_r^{(i)}$ is the i-th column of $A^{(i)}$. Operation $\circ$ represents the outer product of vectors as defined above. Thus each element of $\mathcal{A}$ can be written as $x_{i_1 i_2 \ldots i_N} = \sum_{r=1}^{R} a_{i_1 r}^{(1)} a_{i_2 r}^{(2)} \ldots a_{i_m r}^{(m)}$. For a tensor such as $\mathcal{A}$, the smallest number R in (1) is called the CP rank of $\mathcal{A}$ and the corresponding decomposition is called a CP rank decomposition. Every $A^{(i)}$ is called a factor matrix.

Definition 5 *Let $\mathcal{A} \in F^{I \times I \times \ldots \times I}$ be an mth-order semi-symmetric tensor. A semi-symmetric CP decomposition is defined as*

$$\mathcal{A} = \sum_{r=1}^{R} a_r \circ b_r \circ \ldots \circ b_r \quad (2)$$

in which, a_r and b_r are vectors. The smallest number of R which generates $\mathcal{A}$ exactly, is called the semi-symmetric CP rank of $\mathcal{A}$ and the corresponding decomposition is called a semi-symmetric CP rank decomposition.

The semi-symmetric CP decomposition of a third-order tensor is also called INDSCAL [16, 12]. The semi-symmetric CP decomposition of tensors with orders greater than three can be called higher-order INDSCAL [26]. Also, note that for a vector such as b, we define that $\underbrace{b \circ \ldots \circ b}_{k \text{ times}} := b^{\circ k}$.

If $\{v_1, v_2, ..., v_R\} \in F^q$ is a linearly independent vector set, then there are functionals $\phi_1, ..., \phi_R \in (F^q)^*$ such that

$$\phi_j(v_i) = \delta_{ij} \text{ for } i, j = 1, ..., R.$$

where δ is the Kronecker delta function. Then, suppose that

$$I = \{j_1, ..., j_s\} \subsetneq \{1, ..., m\},$$

$$v_i = a_i^{(j_1)} \circ a_i^{(j_2)} \circ \ldots \circ a_i^{(j_s)}, \quad i = 1, ..., R.$$

Supposing that $\{v_1, v_2, ..., v_R\}$ is a linearly independent set, contracting $\mathcal{A} = \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(m)}$ by functional ϕ_j in I-modes is defined as follows

$$\mathcal{A}._I \phi_k = \sum_{r=1}^{R} \phi_k(v_r) a_r^{(i_1)} \circ a_r^{(i_2)} \circ \ldots \circ a_r^{(i_{m-s})} = a_k^{(i_1)} \circ a_k^{(i_2)} \circ \ldots \circ a_k^{(i_{m-s})},$$

where $\{i_1, i_2, ..., i_{m-s}\} = \{1, ..., m\} \setminus I$. If $\mathcal{A}$ is symmetric then $\mathcal{A}._I \phi_k$ is symmetric. If $\mathcal{A}$ is semi-symmetric and $1 \in I$, then $\mathcal{A}._I \phi_k$ is symmetric, if $1 \notin I$, then $\mathcal{A}._I \phi_k$ is semi-symmetric.

Proposition 2 *Let $\mathcal{A} \in F^{I_1 \times I_2 \times \ldots \times I_m}$ be an mth-order tensor with decomposition (1) and P be a $J \times I_k$ matrix. Then,*

$$\mathcal{A} \times_k P = \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(k-1)} \circ P a_r^{(k)} \circ a_r^{(k+1)} \circ \ldots \circ a_r^{(m)}.$$

Proof. It follows from the linearity of the nth-mode tensor-matrix product. □

Proposition 3 *Let $\mathcal{A}$ be a semi-symmetric mth-order n-dimensional tensor with decomposition (1). Then, there is always a semi-symmetric CP decomposition for $\mathcal{A}$.*

Proof. This proposition can be proved by using the fact that the set of tensors $a \circ T$ for vectors a and symmetric tensors T spans the ambient space. □

3. Rank decomposition and semi-symmetric rank decomposition

In this section, we prove that for a semi-symmetric tensor, the CP rank is equal to the CP semi-symmetric rank under some assumptions. This is analogous to the Comon's conjecture for symmetric tensors [28, 9] and actually we extend results given in [28, 9] for semi-symmetric tensors.

Since for an arbitrary tensor, the semi-symmetric tensor can be easily calculated and it contains many features of its mother tensor, investigating the properties of semi-symmetric tensors can help us to analyse tensors better. In addition, the decomposition of semi-symmetric tensors (especially third-order tensors) has applications, too [8, 25].

We begin by stating results and lemmas which are useful for proving the final theorems. The proof of the following proposition can be found in [17, p.68, Proposition 3.1.3.1.].

Proposition 4 *[17, p.68, Proposition 3.1.3.1] Let $A^{(1)} \circ A^{(2)} \circ \cdots \circ A^{(m)}$ ($m > 2$ and $A^{(i)}, 1 \leq i \leq m$ is a finite-dimensional vector space) be the set of tensors that can be decomposed in the form of $\sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(m)}$ where $a_r^{(i)}$ is a member of $A^{(i)}$. Let $\mathcal{A} \in A^{(1)} \circ A^{(2)} \circ \cdots \circ A^{(m)}$ have rank h. If $\mathcal{A} \in B^{(1)} \circ B^{(2)} \circ \cdots \circ B^{(m)}$ where $B^{(i)} \subseteq A^{(i)}$ with at least one inclusion proper, then any decomposition $\mathcal{A} = \sum_{r=1}^{\rho} a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(m)}$ with some $a_j^{(s)} \notin B^{(s)}$ has $\rho > h$.*

Lemma 1 *Let $\mathcal{A} \in S_1^m(F^n)$ and let (1) be a CP rank decomposition of $\mathcal{A}$ where $R \geq 2$. If $L_j := \mathrm{span}\{a_1^{(j)}, \ldots, a_R^{(j)}\}$ for $j \in \{2, 3, \ldots, m\}$, then for every $i \in \{1, \ldots, R\}$ and $k \in \{2, \ldots, m\}$, $a_i^{(k)} \in L_j$.*

Proof. This lemma is a direct consequence of Proposition 4 □

Lemma 2 *Let $\mathcal{A} \in S_1^m(F^n)$ and let (1) be a CP rank decomposition of $\mathcal{A}$ where $R \geq 2$. Then there exists no $k \in \{2, \ldots, m\}$ such that $a_i^{(k)} \sim a_j^{(k)}$ for every $i, j \in \{1, 2, \ldots, R\}$.*

Proof. Suppose there exists an index $k \in \{2, \ldots, m\}$ such that $a_{i_1}^{(k)} \sim a_{i_2}^{(k)}$ for every $i_1, i_2 \in \{1, 2, \ldots, R\}$. We define $L_k := \mathrm{span}\{a_1^{(k)}, \ldots, a_R^{(k)}\}$, then by assumption, the dimension of L_k is one. By Lemma 1, for every $i \in \{1, \ldots, R\}$ and $j \in \{2, \ldots, m\}$, $a_i^{(j)} \in L_k$. Therefore, for every $i \in \{1, \ldots, R\}$ and $j \in \{2, \ldots, m\}$, we have $a_i^{(j)} \sim a_1^{(k)}$ and

$$\mathcal{A} = \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(m)} = \sum_{r=1}^{R} c_r a_r^{(1)} \circ a_1^{(k)} \circ \ldots \circ a_1^{(k)} = \left(\sum_{r=1}^{R} c_r a_r^{(1)}\right) \circ a_1^{(k)} \circ \ldots \circ a_1^{(k)}$$

where each $c_i, i \in \{1, 2, \ldots, n\}$ is a scalar. This is a contradiction to $R \geq 2$, thus the proof is completed. □

The next lemma is the first step for establishing when the semi-symmetric CP rank decomposition coincides with the CP rank decomposition.

Lemma 3 *If a tensor $\mathcal{A}$ in $S_1^m(F^n)$ ($m \geq 3$) has a CP rank of 1, then every CP rank decomposition is a semi-symmetric CP rank decomposition of $\mathcal{A}$.*

Proof. Since the CP rank decomposition of $\mathcal{A}$ is 1, then

$$\mathcal{A} = a^{(1)} \circ a^{(2)} \circ \ldots \circ a^{(m)}.$$

Since $\mathcal{A}$ is semi-symmetric, then fixing all indices of $\mathcal{A}$ but two of them (of which none is the first index) results in having a symmetric matrix, therefore $a_r^{(2)} \sim a_r^{(i)}$ for $i = 2, 3, \ldots, m$ and thus the CP rank decomposition is a semi-symmetric CP rank decomposition of $\mathcal{A}$.

□

Lemma 4 *Suppose $\mathcal{A}$ is a tensor in $S_1^m(F^n)$ with a CP rank of 2. If $m > 3$, then every CP rank decomposition is a semi-symmetric CP rank decomposition of $\mathcal{A}$. If $m = 3$, then the semi-symmetric CP rank is also 2.*

Proof. We have

$$\mathcal{A} = \sum_{r=1}^{2} a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(m)}.$$

For $m > 3$, from Lemma 2, we know that $a_1^{(k)} \nsim a_2^{(k)}$ for $k \in \{2, 3, ..., m\}$, therefore $\{a_1^{(k)}, a_2^{(k)}\}$ is a linearly independent set for every $k \in \{2, 3, ..., m\}$. Fixing a $k \in \{2, 3, ..., m\}$, there are functionals ϕ_1, ϕ_2 such that $\phi_i(a_1^{(k)}) = \delta_{ik}$, then we have

$$\begin{aligned}\mathcal{A}._k\phi_i &= \sum_{r=1}^{2} \phi_i(a_r^{(k)}) a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(k-1)} \circ a_r^{(k+1)} \circ \ldots \circ a_k^{(m)} \\ &= a_i^{(1)} \circ a_i^{(2)} \circ \ldots \circ a_i^{(k-1)} \circ a_i^{(k+1)} \circ \ldots \circ a_i^{(m)}\end{aligned}$$

which is a semi-symmetric rank-1 tensor. We deduce from the first part of proof that $a_i^{(j_1)} \sim a_i^{(j_2)}$ for $j_1, j_2 \in \{2, ..., m\}$ and $j_1, j_2 \neq k$. Repeating the same procedure for a different k, we can achieve that $a_i^{(j_1)} \sim a_i^{(j_2)}$ for $j_1, j_2 \in \{2, ..., m\}$.

For $m = 3$, if $\mathrm{Dim}\left(\mathrm{Span}\left(\left\{a_1^{(1)}, a_2^{(1)}\right\}\right)\right) = 1$, then

$$\mathcal{A} = \sum_{r=1}^{2} a_r^{(1)} \circ a_r^{(2)} \circ a_r^{(3)} = a_r \circ (\sum_{r=1}^{2} a_r^{(2)} \circ a_r^{(3)}).$$

Since $(\sum_{r=1}^{2} a_r^{(2)} \circ a_r^{(3)})$ is a symmetric matrix and we know that the rank and the symmetric rank for a symmetric matrix are equal [28], therefore the semi-symmetric rank is two.

If $\mathrm{Dim}\left(\mathrm{Span}\left(\left\{a_1^{(1)}, a_2^{(1)}\right\}\right)\right) = 2$, then $a_1^{(1)}$ and $a_2^{(1)}$ are linearly independent. Thus, there are functionals ϕ_1, ϕ_2 such that $\phi_i(a_1^{(k)}) = \delta_{ik}$. The rest of proof is similar to that of $m > 3$ case. $\square$

The next two theorems establish one of our main results.

Theorem 1 *If $\mathcal{A} \in S_1^m(F^n)$ $(m \geq 3)$ and the CP rank of $\mathcal{A}$ is equal to m, then the semi-symmetric CP rank of $\mathcal{A}$ is also m.*

Proof. Suppose (1) is the CP rank decomposition of $\mathcal{A}$. We define $E_2 = \mathrm{Dim}\left(\mathrm{Span}\left(\left\{a_1^{(2)}, a_2^{(2)}, ..., a_R^{(2)}\right\}\right)\right)$. The following cases are possible.

By Lemma 1 and 2, $E_2 = 1$ cannot occur. Thus we have the possible cases (i) $E_2 \geq 3$ and (ii) $E_2 = 2$.

Case (i): By [28, Lemma 3.1], the set $\left\{a_r^{(2)} \circ a_r^{(3)} \circ \ldots \circ a_r^{(m)} | r = 1, ..., R\right\}$ is linearly independent. Therefore, using [28, Lemma 3.5], there is a $j_3 \in \{3, ..., m\}$ such that

$$E_3 = \mathrm{Dim}\left(\mathrm{Span}\left(\left\{a_1^{(2)} \circ a_1^{(j_3)}, a_2^{(2)} \circ a_2^{(j_3)}, ..., a_R^{(2)} \circ a_R^{(j_3)}\right\}\right)\right) > E_2.$$

Continuing in this manner, we can find an ordered index set $I_s = \{2, j_3, ..., j_s\}$ such that $m = R = E_s > E_{s-1} > \cdots > E_2 \geq 3$ and $E_k \geq k+1$, $k = 2, ..., s$ and $s < R$, where

$$E_k = \mathrm{Dim}\left(\mathrm{Span}\left(\left\{a_i^{(2)} \circ a_i^{(j_3)} \circ \cdots \circ a_i^{(j_k)} | i = 1, 2, ..., R\right\}\right)\right), \ k = 2, ..., s.$$

From $E_s \geq s+1$, we know that $s \leq R - 1 = m - 1$. We define $\bar{I}_s := \{2, ..., m\} \setminus I_s = \{j_{s+1}, ..., j_m\}$, define

$$\mathcal{B}_{i,k}^{(1)} := a_i^{(2)} \circ a_i^{(j_3)} \circ \cdots \circ a_i^{(j_k)}, \ \mathcal{B}_{i,k}^{(2)} := a_i^{(j_{k+1})} \circ a_i^{(j_{k+2})} \circ \cdots \circ a_i^{(j_m)},$$

for $k = 2, ..., s$. Then, we can write,

$$\mathcal{A} = \sum_{i=1}^{R} a_i^{(1)} \circ \mathcal{B}_{i,s}^{(1)} \circ \mathcal{B}_{i,s}^{(2)}.$$

By construction, $\left\{\mathcal{B}_{1,s}^{(1)}, ..., \mathcal{B}_{R,s}^{(1)}\right\}$ is linearly independent. Therefore, there exist functionals $\{\phi_1, ..., \phi_R\}$ such that they are dual for $\left\{\mathcal{B}_{1,s}^{(1)}, ..., \mathcal{B}_{R,s}^{(1)}\right\}$. Contracting $\mathcal{A}$ by functional ϕ_j in I_s-modes gives

$$\mathcal{A}._{I_s}\phi_j = \sum_{i=1}^{R} a_i^{(1)} \circ \phi_j(\mathcal{B}_{i,s}^{(1)}) \mathcal{B}_{i,s}^{(2)} = a_j^{(1)} \circ \mathcal{B}_{j,s}^{(2)}.$$

It is clear that $a_j^{(1)} \circ \mathcal{B}_{j,s}^{(2)}$ is a semi-symmetric tensor. Using Lemma 3, we have $a_i^{(j_t)} \sim a_i^{(j_m)}$ for $t = s+1, ..., m$. Thus, for $i = 1, 2, ..., R$, there exists a $y_i \in F^n$ such that $\mathcal{B}_{i,s}^{(2)} = \beta_i y_i^{\circ(m-s)}$ where β_i is a scalar. Using this, we can write

$$\mathcal{A} = \sum_{i=1}^{R} \beta_i a_i^{(1)} \circ \mathcal{B}_{i,s}^{(1)} \circ y_i^{\circ(m-s)} = \sum_{i=1}^{R} \beta_i a_i^{(1)} \circ \mathcal{B}_{i,s-1}^{(1)} \circ a_i^{(j_s)} \circ y_i^{\circ(m-s)}$$

which is a CP rank decomposition. From [28, Lemma 3.1],

$$\left\{a_i^{(1)} \circ \mathcal{B}_{i,s-1}^{(1)} \circ y_i^{\circ(m-s)} | i = 1, 2, ..., R\right\}$$

is linearly independent. Using [28, Lemma 3.5], we can deduce that

$$\left\{a_i^{(1)} \circ \mathcal{B}_{i,s-1}^{(1)} \circ y_i^{\circ\left(R-E_{s-1}\right)} | i = 1, 2, ..., R\right\}$$

is linearly independent. Then, we can find the set of functionals $\{\psi_1, ..., \psi_r\}$ such that $\psi_i \left(a_j^{(1)} \circ \mathcal{B}_{j,s-1}^{(1)} \circ y_j^{\circ\left(R-E_{s-1}\right)}\right) = \delta_{ij}$ for $i, j = 1, ..., R$. Therefore,

$$\begin{aligned}
\mathcal{A}._{\left(I_{s-1}\cup\{j_{s+1}\}\right)}\psi_i &= \sum_{j=1}^{R}\psi_i \left(a_j^{(1)} \circ \mathcal{B}_{j,s-1}^{(1)} \circ y_j^{\circ\left(R-E_{s-1}\right)}\right) y_j^{\circ\left(m-s-(R-E_{s-1})\right)} \circ a_j^{(j_s)} \\
&= y_i^{\circ\left(m-s-(R-E_{s-1})\right)} \circ a_i^{(j_s)}
\end{aligned}$$

which is a symmetric tensor of rank one. Thus, we have $a_i^{(j_s)} \sim y_i$, then

$$\mathcal{A} = \sum_{i=1}^{R}\gamma_i a_i^{(1)} \circ \mathcal{B}_{i,s-1}^{(1)} \circ y_i^{\circ(m-s+1)}$$

where γ_i is a scalar for $i = 1, ..., R$. In a similar manner, we can repeat this process from $s-1$ down to 3, then we have

$$\mathcal{A} = \sum_{i=1}^{R}\delta_i a_i^{(1)} \circ a_i^{(2)} \circ y_i^{\circ(m-2)} = \sum_{i=1}^{R}\delta_i a_i^{(1)} \circ y_i^{\circ(m-2)} \circ a_i^{(2)}.$$

Considering $s = 2$, since $\left\{a_i^{(1)} \circ y_i^{\circ(m-2)} | i = 1, 2, ..., R\right\}$ is linearly independent, we can repeat the process and achieve that $a_k^{(2)} \sim y_k$ and therefore, the CP decomposition is semi-symmetric.

Case (ii): We define $d = \mathrm{Dim}\left(\mathrm{Span}\left(\left\{a_1^{(1)}, a_2^{(1)}, ..., a_R^{(1)}\right\}\right)\right)$, if $d \geq 3$, for $m = 3$, we have that $\{a_1^{(1)}, a_2^{(1)}, a_3^{(1)}\}$ are linearly independent and the rest of proof is similar to that of Lemma 4. If $m > 3$, similar to case $E_2 \geq 3$, there exists an index set $I_s = \{1, j_2, ..., j_s\}$ such that $m = R = E'_s > E'_{s-1} > \cdots > E'_2 > d \geq 3$ and $E'_k \geq k+2, k = 2, ..., s$ and $s < R-1$, where

$$E'_k = \mathrm{Dim}\left(\mathrm{Span}\left(\left\{a_i^{(1)} \circ a_i^{(j_2)} \circ \cdots \circ a_i^{(j_k)} | i = 1, 2, ..., R\right\}\right)\right), \; k = 2, ..., s.$$

It can be seen that $s \leq m-2$, the rest of the proof is similar to $E_2 \geq 3$ case (also see [28, Lemma 4.1]).

Suppose $d = 1$, then there exists a vector a such that $a_i^{(1)} \sim a$ for $i = 1, 2, ..., R$. We define

$$\mathcal{B} := \sum_{r=1}^{m} a_r^{(2)} \circ ... \circ a_r^{(m)}.$$

Since $\mathcal{A}$ is semi-symmetric, $\mathcal{B}$ is symmetric and by definition, the CP rank of $\mathcal{B}$ is less than or equal to m. Using the proof given for [28, Theorem 4.2], there exists a symmetric binary tensor Γ such that the CP rank of $\mathcal{B}$ and Γ are equal. From [28, Lemma 3.11], we know that the CP rank of Γ is less than or equal to $m-1$. Therefore the CP rank of $\mathcal{B}$ is $p \leq m-1$, then

$$\begin{aligned}
\mathcal{B} &= \sum_{r=1}^{m} a_r^{(2)} \circ ... \circ a_r^{(m)} = \sum_{r=1}^{p} b_r^{(2)} \circ ... \circ b_r^{(m)} \Longrightarrow \\
\mathcal{A} &= \sum_{r=1}^{m} a \circ a_r^{(2)} \circ ... \circ a_r^{(m)} = a \circ \left(\sum_{r=1}^{m} a_r^{(2)} \circ ... \circ a_r^{(m)}\right) = a \circ \mathcal{B} \\
&= a \circ \left(\sum_{r=1}^{p} b_r^{(2)} \circ ... \circ b_r^{(m)}\right) = \sum_{r=1}^{p} a \circ a_r^{(2)} \circ ... \circ a_r^{(m)}
\end{aligned}$$

which is a contradiction to m being the minimal rank. Therefore, d cannot equal to one.

If $d = 2$, from Lemma 1, since $E_2 = 2$, we have

$$\mathrm{Dim}\left(\mathrm{Span}\left(\left\{a_j^{(i)} | i \in \{2, ..., m\}, j \in \{1, 2, ..., R\}\right\}\right)\right) = 2.$$

Then, there exist two linearly independent vectors p_1, p_2 such that for $i \in \{2, ..., m\}$, $j \in \{1, 2, ..., R\}$, $a_j^{(i)} = Pb_j^{(i)}$ where $P = [p_1, p_2]$. Also, from $d = 2$, we know there are two linearly independent vectors q_1, q_2 such that for $i \in \{2, ..., m\}$, $a_j^{(1)} = Qb_j^{(1)}$ where $Q = [q_1, q_2]$. Therefore, we have

$$\begin{aligned}\mathcal{A} &= \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(m)} \\ &= \sum_{r=1}^{R} Qb_r^{(1)} \circ Pb_r^{(2)} \circ \ldots \circ Pb_r^{(m)} \\ &= (\sum_{r=1}^{R} b_r^{(1)} \circ b_r^{(2)} \circ \ldots \circ b_r^{(m)}) \times_m P \times_{m-1} P \ldots \times_2 P \times_1 Q.\end{aligned}$$

Let

$$\mathcal{L} = \sum_{r=1}^{R} b_r^{(1)} \circ b_r^{(2)} \circ \ldots \circ b_r^{(m)}$$

Then, $\mathcal{L}$ is an mth-order 2-dimensional tensor. We want to prove that the CP rank and the semi-symmetric CP rank of $\mathcal{L}$ and $\mathcal{A}$ are equal.

If a tensor is multiplied in any mode by an invertible square matrix, then the CP rank won't change (see [6, Theorem 3.10 and Theorem 3.12]). Thus, if $\mathcal{A}$ is multiplied in any mode invertible square matrices, then the semi-symmetric CP rank won't change.

Obviously, P and Q are $n \times 2$ rank-two matrices, therefore there are two $n \times n$ square invertible matrices C and D such that

$$CP = DQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} = [e_1, e_2].$$

Thus, using Proposition 2, we have

$$\begin{aligned}\mathcal{A} \times_m C \times_{m-1} C \ldots \times_2 C \times_1 D &= \mathcal{L} \times_m CP \times_{m-1} CP \ldots \times_2 CP \times_1 DQ \\ &= \mathcal{L} \times_m [e_1, e_2] \times_{m-1} [e_1, e_2] \ldots \times_2 [e_1, e_2] \times_1 [e_1, e_2] \\ &= \sum_{r=1}^{R} [e_1, e_2] b_r^{(1)} \circ [e_1, e_2] b_r^{(2)} \circ \ldots \circ [e_1, e_2] b_r^{(m)} \\ &= \sum_{r=1}^{R} \begin{bmatrix} (b_r^{(1)})_1 \\ (b_r^{(1)})_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \circ \begin{bmatrix} (b_r^{(2)})_1 \\ (b_r^{(2)})_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \circ \ldots \circ \begin{bmatrix} (b_r^{(m)})_1 \\ (b_r^{(m)})_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.\end{aligned}$$

Therefore the CP rank of $\mathcal{L}$ is equal to CP rank of $\mathcal{A}$. The same argument can be used for the semi-symmetric CP rank of $\mathcal{L}$ and we can deduce that this rank is also equal to that of $\mathcal{A}$.

By Proposition 3, there exists a semi-symmetric CP decomposition for $\mathcal{L}$, that is, there exists vectors a_r and b_r for $r = 1, ..., T$ such that

$$\mathcal{L} = \sum_{r=1}^{T} a_r \circ b_r^{\circ m-1}.$$

From [28, Lemma 3.1], we see that $\{b_r^{\circ m-1} | r = 1, 2, ..., T\}$ is a linearly independent set. On other hand, for every $r = 1, 2, ..., T$, $b_r^{\circ m-1}$ is a symmetric $(m-1)$th-order 2-dimensional tensor and therefore it has, at maximum, m distinct members (every mth-order n-dimensional symmetric tensors have $\begin{pmatrix} n+m-1 \\ m \end{pmatrix}$ distinct components [1]). Thus, $T \leq m$ which means that the semi-symmetric CP rank of $\mathcal{A}$ is less than/or equal to m and since the CP rank of $\mathcal{A}$ is m, semi-symmetric CP rank of $\mathcal{A}$ is equal to m. $\square$

Theorem 2 *If $\mathcal{A} \in S_1^m(F^n)$ $(m \geq 3)$ and the CP rank of $\mathcal{A}$ is $R < m$, then the CP rank of $\mathcal{A}$ is equal to its semi-symmetric CP rank.*

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Proof. There are two cases (i) $R \leq 2$ and (ii) $R > 2$.

Case (i): For the rank less than 3, from Lemma 3 and Lemma 4, we have the result.

Case (ii): Suppose that (1) is the CP rank decomposition of $\mathcal{A}$ with $R > 2$. We define

$$E_1 = \mathrm{Dim}\left(\mathrm{Span}\left(\left\{a_1^{(1)}, a_2^{(1)}, \dots, a_R^{(1)}\right\}\right)\right).$$

If $E_1 = 1$, then there exists a vector a such that $a_i^{(1)} \sim a$ for $i = 1, 2, \dots, R$. In this case, we have

$$\mathcal{A} = \sum_{r=1}^{R} a \circ a_r^{(2)} \circ \dots \circ a_r^{(m)} = a \circ \left(\sum_{r=1}^{R} a_r^{(2)} \circ \dots \circ a_r^{(m)}\right).$$

Since $\mathcal{A}$ is semi-symmetric, then $\mathcal{B} := \sum_{r=1}^{R} a_r^{(2)} \circ \dots \circ a_r^{(m)}$ is a symmetric tensor. Note that the CP rank of $\mathcal{B}$ should be R, because otherwise we have a contradiction to the fact that R is the minimal CP rank of $\mathcal{A}$. Also, by [28, Theorem 3.8] and [28, Theorem 4.2], since $R \leq m-1$, then there exists a symmetric CP-decomposition of $\mathcal{B}$ of rank R, then there exist vectors b_r, $r \in \{1, \dots, R\}$ such that

$$\mathcal{A} = \sum_{r=1}^{R} a \circ b_r^{\circ m-1}$$

which proves the theorem in this case.

If $E_1 \geq 2$, then the proof is similar to the proof of previous theorem, part $E_2 \geq 3$, the only difference is, we just need to follow it from $s-1$ down to 2, since the rank is smaller in this theorem. Also, see the symmetric case in [28, Theorem 3.8]. □

Corollary 1 *For a third-order n-dimensional semi-symmetric tensor with CP rank equal or less than three, semi-symmetric CP rank is the same as CP rank.*

Therefore, we could obtain a result which states that under certain conditions dependent on the order of tensor, the CP rank and semi-symmetric rank are equal.

We next establish results analogous to that given for symmetric tensors in [9, 13]. This time, equality holds under some assumptions on the tensor dimension sizes.

Lemma 5 *Let $\{b_r | r = 1, 2, \dots, R\}$ be a linearly independent set. If tensor $\mathcal{A} \in S_1^m(F^n)$ $(m > 1)$ is defined as follows*

$$\mathcal{A} = \sum_{r=1}^{R} a_r \circ b_r^{\circ m-1}$$

with $a_r \neq 0$ for every r, then the semi-symmetric CP rank of $\mathcal{A}$ is R.

Proof. For $m = 2$, the result is obvious. Suppose the CP rank of $\mathcal{A}$ is $s < R$, then there exist c_r, d_r for $r = 1, \dots, s$ such that

$$\mathcal{A} = \sum_{r=1}^{R} a_r \circ b_r^{\circ m-1} = \sum_{r=1}^{s} c_r \circ d_r^{\circ m-1}.$$

There exist functionals $\{\phi_1, \dots, \phi_R\}$ such that $\phi_i(b_j) = \delta_{ij}$. Let us choose an arbitrary scalar i from $\{1, 2, \dots, R\}$. We know there exist at least a $j \in \{1, \dots, n\}$ such that $(a_i)_j \neq 0$ (because otherwise we have $a_i \equiv 0$), then

$$\mathcal{A}(j, :, \dots, :) = \sum_{r=1}^{R} (a_r)_j b_r^{\circ m-1} = \sum_{r=1}^{s} (c_r)_j d_r^{\circ m-1}.$$

Contracting the above equation by functional ϕ_i in the first $m-2$ modes gives the following equation:

$$(a_i)_j b_i = \sum_{r=1}^{s} (c_r)_j \alpha_r d_r.$$

Since $(a_i)_j \neq 0$, this equation implies that $b_i \in \mathrm{span}\{d_1, \dots, d_s\}$. Since i was chosen arbitrary, then $\mathrm{span}\{b_1, \dots, b_R\} \subseteq \mathrm{span}\{d_1, \dots, d_s\}$. Since $s < R$ and due to linear independence of $\mathrm{span}\{b_1, \dots, b_R\}$, this is a contradiction. Therefore, the semi-symmetric CP rank of $\mathcal{A}$ is R. □

We note that a similar result to Lemma 5 is mentioned in [17, Exercise 3.1.3.3]. Also, for $m > 3$, using Kruskal's result [16], we can see that the decomposition given in Lemma 5 is unique.

In the following, we use the term "generically", a property holds generically if it holds everywhere but for a set of Lebesgue measure zero [10]. Also, see [14], [9, Section 4.4 and Lemma 5.2] for more information about this lemma and the concept of genericity.

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Lemma 6 *Let $\mathcal{A} \in S_1^m(F^n)$ and*

$$\mathcal{A} = \sum_{r=1}^{R} a_r \circ b_r \circ \ldots \circ b_r$$

be a semi-symmetric CP rank decomposition of $\mathcal{A}$ with $R \leq n$. Then,

$$\{b_r | r = 1, 2, \ldots, R\}$$

is generically a linearly independent set.

Proof. The proof is similar to the that of [9, Lemma 5.2] given for symmetric tensors. We should just change the function given in the proof of [9, Lemma 5.2] to the following function

$$E_A : F^{n\times R} \to S_1^m(F^n)$$
$$[b_1, \ldots, b_R] \longmapsto \sum_{r=1}^{R} a_r \circ b_r \circ \ldots \circ b_r$$

where $A = [a_1, \ldots, a_R]$ and obviously f is a polynomial map. The rest follows from the proof of [9, Lemma 5.2]. □

Theorem 3 *Let $\mathcal{A} \in S_1^m(F^n)$. If the semi-symmetric CP rank of $\mathcal{A}$ is equal to or less than n, then the CP rank is equal to the CP semi-symmetric rank generically.*

Proof. Suppose R and E_{ss} are the CP rank and the CP semi-symmetric rank, respectively. We have,

$$\sum_{r=1}^{R} c_r^{(1)} \circ c_r^{(2)} \circ \ldots \circ c_r^{(m)} = \sum_{r=1}^{E_{ss}} a_r \circ b_r \circ \ldots \circ b_r. \tag{3}$$

From [28, Lemma 3.1], we know that $\left\{a_r \circ b_r^{\circ(m-2)} | r = 1, 2, \ldots, E_{ss}\right\}$ is a linearly independent set. Therefore, there exist functionals $\{\phi_1, \ldots, \phi_{E_{ss}}\}$ such that $\phi_i(a_j \circ b_j^{\circ(m-2)}) = \delta_{ij}$. Fix an $i \in \{1, 2, \ldots, E_{ss}\}$. Contracting both sides of (3) by ϕ_i yields

$$\sum_{r=1}^{R} \gamma_{ij} c_r^{(m)} = b_i$$

where $\gamma_{ij} = \phi_i(c_j^{(1)} \circ c_j^{(2)} \circ \ldots \circ c_j^{(m-1)})$. Since i was arbitrary, then for $i \in \{1, 2, \ldots, E_{ss}\}$, $b_i \in \mathrm{span}\{c_1^{(m)}, \ldots, c_R^{(m)}\}$. By Lemma 6, $\{b_r | r = 1, 2, \ldots, E_{ss}\}$ is generically a linearly independent set. As

$$\mathrm{span}\{b_1, \ldots, b_{E_{ss}}\} \subseteq \mathrm{span}\{c_1^{(m)}, \ldots, c_R^{(m)}\},$$

then $E_{ss} \leq R$. Obviously, we have also $E_{ss} \geq R$, therefore $E_{ss} = R$. □

Corollary 2 *For a third-order n-dimensional semi-symmetric tensor with CP rank less or equal than n, semi-symmetric CP rank is the same as CP rank generically.*

Remark 1 *The results of this paper are stated for cubic tensors, for a non-cubic tensor, that is a tensor such as $\mathcal{A} \in F^{k\times n\times \ldots \times n}$ where $k \neq n$, if $k > n$ then we can extend this tensor by adding zero elements to this tensor and make it a cubic tensor like $\mathcal{A}' \in F^{k\times k\times \ldots \times k}$, it can be easily seen that every decomposition of $\mathcal{A}$ can become a decomposition of $\mathcal{A}'$ by adding the respective zero elements to decompositions. So, the results of this paper are true for cubic $\mathcal{A}'$ and therefore for $\mathcal{A}$. If $k < n$, we can again by adding zero elements create a cubic tensor in $F^{n\times n\times \ldots \times n}$, the previous argument can be used to show that the results hold for $\mathcal{A}'$ and therefore $\mathcal{A}$. These results are included as the following Corollary.*

Corollary 3 *Let $\mathcal{A} \in F^{k\times n\times \ldots \times n}$ be an mth-order semi-symmetric tensor. Suppose $\mathcal{A} = \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ \ldots \circ a_r^{(m)}$ is the CP rank decomposition of $\mathcal{A}$.*

i) If $R \leq m$, then the rank CP and semi-symmetric rank of $\mathcal{A}$ are equal.
ii) If the semi-symmetric CP rank is less or equal to n, then the CP rank is equal to the CP semi-symmetric rank generically.

4. Conclusions and Future Works

The connection between the CP rank and semi-symmetric rank (higher-order INDSCAL) is investigated in this paper. Under some assumptions on the size (or order) of a tensor, the CP and semi-symmetric CP rank are proved to be equal.

In future work, the relationship between the decomposition of a tensor and the decomposition of its analogous semi-symmetric tensor can be investigated. Since semi-symmetric tensors have a simpler structure and for every tensor there exists a semi-symmetric tensor which inherits many of its properties (like eigenvalues) [18], establishing such a relationship can be especially useful for better understanding the tensor decompositions.

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