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Numerical solution of differential equations of Lane-Emden type by Gegenbauer and rational Gegenbauer collocation methods

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In this paper, we apply the collocation method for solving some classes of Lane-Emden type equations that are determined in interval $[0, 1]$ and semi-infinite domain. We use an orthogonal system of functions, namely Gegenbauer polynomials and introduce the shifted Gegenbauer polynomials and the rational Gegenbauer functions as basis functions in the collocation method for problems in interval $[0, 1]$ and semi-infinite domain, respectively. We estimate that the proposed method has super-linear convergence rate and also investigate the Gegenbauer parameter (α) to get more accurate answers for various Lane-Emden type problems. The comparison between the proposed method and other numerical results shows that the method is efficient and applicable.

Keywords: Gegenbauer polynomials; Rational Gegenbauer functions; Collocation method; Nonlinear ODE; Lane-Emden equations; Astrophysics.

1. Introduction

In this section, we will first have a general introduction about the spectral methods that are used to solve different types of equations. Then we examine the history of the methods used to solve the Lane-Emden equation, which is widely used in the world of astronomy. In this article, we are going to solve the Lane-Emden equation by using one of the common and widely used spectral methods called collocation method.

1.1. A brief introduction to spectral methods

Spectral and pseudospectral methods have been successfully applied in various problems of science and engineering which modeled by differential equations in finite domain. For problems whose solutions are sufficiently smooth, they exhibit exponential rates of convergence/spectral accuracy. We can apply different spectral methods that are used to solve these problems [1, 2, 3]. The most common method is the use of polynomials that are orthogonal, such as the Legendre and Chebyshev polynomials as basis functions in collocation methods [4, 5, 6]. Furthermore, another numerical methods, such as meshless [7, 8] and semi-analytical methods and homotopy analysis method [9], have been used to solve science and engineering problems. In addition to the mentioned methods, recently methods based on machine learning methods, such as kernel-based methods [10, 11] and neural network methods [12] have been used to solve various type of equations. For instance, a method for solving an ordinary differential equation by support vector machine was presented in [13]. A feed forward neural network for solving differential equations of fractional order efficiency, has been proposed in [14]. Using orthogonal polynomials as kernel functions in support vector machine method have been discussed in [15, 16]. Introducing a new neural network as fractional Chebyshev deep neural network (FCDN) for solving differential equations of fractional order by using Chebyshev polynomials has been presented in [17].

On the other hand, many science and engineering problems of current interest are set in unbounded domains. In the context of spectral methods, a number of approaches have been proposed and investigated for treating unbounded domains. The most common one is the use of polynomials that are orthogonal over unbounded domains, such as the Hermite and Laguerre spectral methods [18, 19, 20, 21, 22, 23, 24, 25].

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Guo [26, 27, 28, 29] proposed a method in which the original problem in an unbounded domain is mapped to a problem in a bounded domain and then suitable Jacobi polynomials such as Gegenbauer polynomials are used to approximate the resulting problems. Another approach is to replace the infinite domain with [-L, L] and the semi-infinite interval with [0, L] by choosing L sufficiently large. This method is called domain truncation [30]. Another effective direct approach to solving such problems is based on rational approximations. Christov [31] and Boyd [32, 33] developed some spectral methods for unbounded intervals by applying mutually orthogonal systems of rational functions. Boyd [32] defined a new spectral basis, named rational functions on the semi-infinite interval, by mapping them to Chebyshev polynomials. Guo et al. [34] introduced a new set of rational Legendre functions which are orthogonal to each other in $L^2(0, +\infty)$. He also introduced an orthogonal system on the half line, induced by Jacobi polynomials in [35]. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half-line. Boyd et al. [36] applied the pseudospectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre and mapped Fourier sine methods. Parand et al. [4, 5, 6, 37, 38, 39, 40, 41], applied the spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on rational Tau and collocation methods.

1.2. Lane-Emden equation

Lane-Emden equations are singular initial value problems related to second-order ordinary differential equations (ODEs) which have been used to model several phenomena in mathematical physics and astrophysics. Recently, many analytical methods have been used to solve Lane-Emden equations, however, the main difficulty arises in the singularity of the equations. The methods used to solve this equation include perturbation technique, quasilinearization approach, Adomian decomposition method, Ritz's method, homotopy analysis method, variational iteration method, Lie symmetry approach, spectral and decomposition.

Parand et al. [6, 51, 52, 53, 54] presented two numerical techniques to solve higher ordinary differential equations such as Lane-Emden. Their approach was based on the pseudospectral and Tau methods. Yousefi [59] presented a numerical method for solving the Lane-Emden equations. He converted the Lane-Emden equations to integral equations, using integral operator, and then he applied Legendre wavelet approximations. Recently, also another numerical methods based on the B-spline expansion and the collocation approach [60], the Liapunov-Schmidt reduction and symmetry-breaking bifurcation theory [61], cubic Hermite spline functions [62] and matrix relations of Laguerre polynomials [63] were presented to solve the Lane-Emden type equations.

Recently, another methods also have been applied to solve the Lane-Emden problem. For example, in [77] Boubaker Polynomials Expansion Scheme (BPES) applied to obtain analytical numerical solutions and Bhrawy et al. [78] used shifted Jacobi pseudo-spectral method with BPES to solve this problem and Shen [79] has combined the compactly supported radial basis function (RBF) collocation method and the scaling iterative algorithm to solve Lane-Emden-Fowler equation on various domains. Also, recently, machine learning methods have been used to solve the Lane-Emden problem [80, 81].

In this paper, we want to solve problems such as Lane-Emden equations by using an orthogonal system of functions, namely Gegenbauer polynomials in collocation method. The Gegenbauer polynomials, often called Ultraspherical polynomials, include Legendre and Chebyshev polynomials as special or limiting cases. For solving bounded domain problems, we can apply Gegenbauer polynomials as basis functions in collocation method. But for problems in unbounded domains, we present the spectral method on the half-line by using an orthogonal rational system of functions, namely rational Gegenbauer functions. We Also prove that the proposed method has the super-linear convergence rate and obtain results with good accuracy in comparison with other numerical methods.

The remainder of this paper is organized as follows. Section 2 reviews the desirable properties of Gegenbauer polynomials and rational Gegenbauer functions as well as the error bound for their function approximation. In Section 3, we describe Lane-Emden model which contains the well-known types of equations in bounded and unbounded domains. In Section 4, we apply the Gegenbauer and rational Gegenbauer functions collocation methods to solve some Lane-Emden type equations as the application of theses methods. Finally, Section 5 makes concluding remarks.

2. Gegenbauer functions

In this section, we first introduce Gegenbauer polynomials and exponential Gegenbauer functions and state some of their basic features. Then, we approximate a function using Gaussian integration with Gegenbauer-Gauss points and rational Gegenbauer-Gauss points in the collocation method. The last part is about the discussion of the super-linear convergence for both of shifted and rational Gegenbauer collocation methods.

Here we introduce the following notation for $\gamma = [a, b]$ or $[0, \infty)$

$$L^2_{\varpi}(\gamma) = \{v : \gamma \to \mathbb{R} \mid v \text{ is measurable and } \| v \|_{\varpi} < \infty\}, \tag{1}$$

where

$$\| v\|_{\varpi} = \left(\int_{\gamma} | v(x) |^2 \varpi(x)dx\right)^{\frac{1}{2}}, \tag{2}$$

is the norm induced by the scalar product

$$< u, v >_{\varpi} = \int_{\gamma} u(x)v(x)\varpi(x)dx. \tag{3}$$

2.1. Properties of Gegenbauer polynomials

The Gegenbauer polynomials $G_n^\alpha(y)$ of order α and degree n are defined as follows [88]:

$$G_n^\alpha(y) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{\Gamma(n+\alpha-j)}{j!(n-2j)!\Gamma(\alpha)} (2y)^{n-2j}, \tag{4}$$

where n is an integer, α is a real number greater than $-\frac{1}{2}$ and Γ is the Gamma function.

We have the following generating formula for the Gegenbauer polynomials [89]:

$$(1-2yx+x^2)^{-\alpha} = \sum_{n=0}^{\infty} G_n^\alpha(y) x^n. \tag{5}$$

So $G_n^\alpha(y)$ can be defined as the coefficients of x^n in the expansion of $(1-2yx+x^2)^{-\alpha}$.

These polynomials are a special case of Jacobi polynomials and generalized Legendre and Chebyshev polynomials. The special cases $\alpha = 0, 1, \frac{1}{2}$ correspond to the Chebyshev polynomials of first kind, second kind and Legendre polynomials, respectively.

The Gegenbauer polynomials are orthogonal in the interval $[-1,1]$ with respect to the weight function $\rho(y) = (1-y^2)^{\alpha-\frac{1}{2}}$ where $\alpha > -\frac{1}{2}$. For a fixed α, these polynomials are defined under Eq. (3), i.e.

$$< G_n^\alpha, G_m^\alpha >_\rho = \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)\Gamma^2(\alpha)} \delta_{nm}, \tag{6}$$

where n and m are the degrees of the polynomials and δ_{nm} is the Kronecker delta function [90].

They can be determined by the following recurrence formula [90]:

$$\begin{aligned} &G_0^\alpha(y) = 1, \qquad G_1^\alpha(y) = 2\alpha y, \\ &G_{n+1}^\alpha(y) = \frac{1}{n+1}\left[2y(n+\alpha)G_n^\alpha(y) - (n+2\alpha-1)G_{n-1}^\alpha(y)\right], \quad n \geq 1. \end{aligned} \tag{7}$$

$G_n^\alpha(y)$ is the nth eigenfunction of the singular Sturm-Liouville problem [88]:

$$\sqrt{1-y^2}\frac{d}{dy}\left[\sqrt{1-y^2}\frac{d}{dy}G_n^\alpha(y)\right] - 2\alpha y \frac{d}{dy}G_n^\alpha(y) + n(n+2\alpha)G_n^\alpha(y) = 0. \tag{8}$$

In [88] we find Rodrigues formula for Gegenbauer polynomials

$$(1-y^2)^{\alpha-\frac{1}{2}} G_n^\alpha(y) = \frac{(-2)^n}{n!} \frac{\Gamma(n+\alpha)\Gamma(n+2\alpha)}{\Gamma(\alpha)\Gamma(2n+2\alpha)} \frac{d^n}{dy^n}(1-y^2)^{n+\alpha-\frac{1}{2}}. \tag{9}$$

From the differentiation formula for Gegenbauer polynomials [88]

$$G_n'^\alpha(y) = \frac{d}{dy}G_n^\alpha(y) = 2\alpha G_{n-1}^{(\alpha+1)}(y), \tag{10}$$

and Eqs. (3) and (6), we see that $G_n'^\alpha(y)$ are orthogonal in $[-1,1]$ with $\hat{\rho}(y) = \frac{1}{4\alpha^2}(1-y^2)^{(\alpha+\frac{1}{2})}$ as the weight function, thus,

$$< G_n'^\alpha, G_m'^\alpha >_{\hat{\rho}} = \frac{\pi 2^{-(2\alpha+1)}\Gamma(2\alpha+n+1)}{(n-1)!(n+\alpha)\Gamma^2(\alpha+1)} \delta_{nm}. \tag{11}$$

To approximate $u(y)$ as a function with Gegenbauer polynomials, we have the following expansion:

$$u(y) = \sum_{k=0}^{+\infty} c_k G_k^\alpha(y), \tag{12}$$

with

$$c_k = \frac{< f, G_k^\alpha >_\rho}{\| G_k^\alpha \|_\rho^2}. \tag{13}$$

The c_k's are the expansion coefficients associated with the family $\{G_k^\alpha(y)\}$.

To find the unknown coefficients c_k's, we can use the collocation method with Gegenbauer-Gauss points.

If we want to solve the problem in the $\gamma = [a,b]$ interval, we should use the mapping $\psi(y) = \frac{1}{2}[(b-a)y + (b+a)]$ first, to map Gegegnbauer polynomials into the interval $[a,b]$ and then apply the collocation method with shifted Gegenbauer-Gauss points. Indeed, we have the shifted Gegenbauer (SG) polynomials.

2.2. Properties of rational Gegenbauer functions

The new basis function is recorded as $RG_n^\alpha(x) = G_n^\alpha(y)$, where L is a constant parameter, $y = \frac{x-L}{x+L}$, $y\ in[-1,1]$. The constant parameter L specifies the length scale of the map. Boyd [91] provides guidelines for optimizing L map parameters. $RG_n^\alpha(x)$ is the nth eigenfunction of the singular Sturm-Liouville problem:

$$(x+L)\frac{\sqrt{x}}{L}\frac{d}{dx}\left[(x+L)\sqrt{x}\frac{d}{dx}RG_n^\alpha(x)\right] + \alpha\left(\frac{L^2-x^2}{L}\right)\frac{d}{dx}RG_n^\alpha(x) + n(n+2\alpha)RG_n^\alpha(x) = 0, \tag{14}$$

and satisfies in the following recurrence relation:

$$\begin{aligned} &RG_0^\alpha(x) = 1, \qquad RG_1^\alpha(x) = 2\alpha\frac{x-L}{x+L}, \\ &RG_{n+1}^\alpha(x) = \frac{1}{n+1}\left[2\left(\frac{x-L}{x+L}\right)(n+\alpha)RG_n^\alpha(x) - (n+2\alpha-1)RG_{n-1}^\alpha(x)\right], \quad n \geq 1. \end{aligned} \tag{15}$$

We determine $w(x) = \frac{2L}{(x+L)^2}\left[1-\left(\frac{x-L}{x+L}\right)^2\right]^{\alpha-\frac{1}{2}}$ as a non-negative, integrable, and real-valued weight function for rational Gegenbauer over the interval $\gamma = [0,\infty)$.

Let us denote [92]

$$\rho(y) = (1-y^2)^{\alpha-\frac{1}{2}}, \qquad y = \frac{x-L}{x+L}, \tag{16}$$

hence we have [92]

$$\frac{dy}{dx} = \frac{2L}{(x+L)^2}, \qquad \frac{dx}{dy} = \frac{2L}{(y-1)^2}, \qquad w(x)\frac{dx}{dy} = \rho(y). \tag{17}$$

Thus $\{RG_n^\alpha(x)\}_{n\geq 0}$ denotes a system which is mutually orthogonal under Eq. (3), i.e.

$$< RG_n^\alpha, RG_m^\alpha >_w = \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)\Gamma^2(\alpha)}\delta_{nm}. \tag{18}$$

This system is complete in $L_w^2(\gamma)$. For any function $u \in L_w^2(\gamma)$, the following expansion holds

$$u(x) = \sum_{k=0}^{+\infty} a_k RG_k^\alpha(x), \tag{19}$$

with

$$a_k = \frac{< u, RG_k^\alpha >_w}{\| RG_k^\alpha \|_w^2}. \tag{20}$$

The a_k's are the expansion coefficients associated with the family $\{RG_k^\alpha(x)\}$.

The differentiation formula for rational Gegenbauer can be obtained as:

$$RG_n'^\alpha(x) = \frac{d}{dx}RG_n^\alpha(x) = \frac{4\alpha L}{(x+L)^2}RG_{n-1}^{(\alpha+1)}(x). \tag{21}$$

So, we find that $RG_n'^\alpha(x)$ also are mutually orthogonal in $L_{\hat{w}}^2(\gamma)$ with respect to the weight function $\hat{w}(x) = \frac{(x+L)^2}{8L\alpha^2}\left[1-\left(\frac{x-L}{x+L}\right)^2\right]^{\alpha+\frac{1}{2}}$. Hence

$$< RG_n'^\alpha, RG_m'^\alpha >_{\hat{w}} = \frac{\pi 2^{-(2\alpha+1)}\Gamma(2\alpha+n+1)}{(n-1)!(n+\alpha)\Gamma^2(\alpha+1)}\delta_{nm}. \tag{22}$$

The same as Gegenbauer polynomials, the special cases of the rational Gegenbauer functions are rational Legendre functions and also rational Chebyshev functions, that were introduced by Guo [34, 92].

Remark 1 *It is notable that, in addition to the rational mapping, other mapping $y = \phi(x)$, in which $\phi : [0,\infty) \to [-1,1]$, can be used to generate the new basic functions. Two of them are mentioned below, which can be used in future research. Here too, L is a constant parameter.*

- *Exponential mapping: $y = 1 - 2e^{\frac{-x}{L}}$*
- *Logarithmic mapping: $y = -1 + 2\tanh\left(\frac{x}{L}\right)$*

For these mappings, you can proceed in the same way as the process mentioned for rational functions and use them for interpolation approximation by collocation method.

2.3. Gegenbauer and rational Gegenbauer interpolation approximation

Authors of [93, 94] introduced Gauss integration. Let $\mathcal{P}_N$ be the space of polynomials of degree less than or equal to N and y_j, $j = 0, 1, \ldots, N$, be the $N+1$ roots of the polynomial $G^{\alpha}_{N+1}(y)$. These points are known as Gegenbauer-Gauss points.

By Gaussian integration we have:

$$\int_{-1}^{1} u(y)\rho(y)dy = \sum_{j=0}^{N} u(y_j)\rho_j \quad \forall u \in \mathcal{P}_{2N}, \tag{23}$$

where ρ_j as corresponding Christoffel numbers are defined as [88]:

$$\rho_j = \frac{2^{2-2\alpha}\pi\Gamma(N+1+2\alpha)}{(N+1)!\Gamma^2(\alpha)} \frac{1}{(1-y_j{}^2)[\frac{d}{dy}G^{\alpha}_{N+1}(y_j)]^2}. \tag{24}$$

Here we define the operator $I_N : L^2_w[-1,1] \to \mathcal{P}_N$ by

$$I_N u(y) = \sum_{k=0}^{N} c_k G^{\alpha}_k(y). \tag{25}$$

$I_N u$ is the orthogonal projection of u upon $\mathcal{P}_N$ with respect to the inner product (3) and the norm (2) with $\gamma = [-1, 1]$. Thus by the orthogonality of the Gegenbauer polynomials, we have [27]

$$< I_N u - u, G^{\alpha}_i >_{\rho} = 0 \quad \forall G^{\alpha}_i \in \mathcal{P}_N. \tag{26}$$

We note that all of the above statements can be generalized to the case when $\gamma = [a, b]$ by the shifted Gangbaner polynomials.

In [34, 92] Guo introduced rational Legendre-Gauss points and rational Chebyshev-Gauss points, respectively. Now we want to define rational Gegenbauer-Gauss interpolation. Let

$$\mathcal{RG}^{\alpha}_N = span\{RG^{\alpha}_0, RG^{\alpha}_1, ..., RG^{\alpha}_N\}, \tag{27}$$

and

$$x_j = L\frac{1+y_j}{1-y_j} \quad j = 0, 1, \ldots, N, \tag{28}$$

which are called as rational Gegenbauer-Gauss nodes and y_j are Gegenbauer-Gauss points. In fact, these points are zeros of the function $RG^{\alpha}_{N+1}(x)$. Using Gauss integration, we have:

$$\begin{aligned} \int_0^{\infty} u(x)w(x)dx &= \int_{-1}^{1} u\left(L\frac{1+y}{1-y}\right)\rho(y)dy \\ &= \sum_{j=0}^{N} u(x_j)w_j \quad \forall u \in \mathcal{RG}^{\alpha}_{2N}, \end{aligned} \tag{29}$$

where

$$w_j = \frac{2^{2-2\alpha}\pi\Gamma(N+1+2\alpha)}{(N+1)!\Gamma^2(\alpha)} \frac{2L}{x_j(x_j+L)^2[\frac{d}{dx}RG^{\alpha}_{N+1}(x_j)]^2}, \tag{30}$$

are the corresponding weights with the $N+1$ rational Gegenbauer-Gauss nodes which can be obtained from Eqs. (24) and (28).

$P_N u$ stands for the interpolating function of a smooth function u on a semi-infinite interval. It is a component of the $\mathcal{RG}^{\alpha}_N$ structure and is described as

$$P_N u(x) = \sum_{k=0}^{N} a_k RG^{\alpha}_k(x). \tag{31}$$

$P_N u$ is the orthogonal projection of u upon $\mathcal{RG}^{\alpha}_N$ with respect to the inner product (3) and the norm (2) with $\gamma = [0, \infty)$. Thus by the orthogonality of the rational Gegenbauer functions, we have [27]

$$< P_N u - u, RG^{\alpha}_i >_w = 0 \quad \forall RG^{\alpha}_i \in \mathcal{RG}^{\alpha}_N. \tag{32}$$

We take into account the residual function, $Res(x)$, when the expansion is inserted into the governing equation in order to use a collocation method. The a_k's must be chosen to satisfy the boundary constraints while producing the residual zero at the greatest number of (appropriately chosen) spatial places.

2.4. Convergence of the function approximation

We describe this subsection for both of shifted and rational Gegenbauer collocation methods. for approximating any function $u(x) \in L^2_w(\gamma)$ where $\gamma = [0, \zeta]$ and $\zeta = 1$ or ∞, we can determine the error of the solution as follows if the operator IP_N considered as I_N or P_N and ϖ as their corresponding weight w or ρ, respectively:

$$e_N = \| IP_N u - u \|^2_{\varpi} . \tag{33}$$

According to the orthogonality and completeness of the systems $\{SG^{\alpha}_j(x)\}_{j\geq 0}$ and $\{RG^{\alpha}_j(x)\}_{j\geq 0}$ the following theorem, whose proof approach is almost the same as the proof of theorem 1 in [95], can be achieved.

Theorem 1 *Let $IP_N u(x)$ is the approximation of $u(x) \in L^2_{\varpi}(\gamma)$, ($\gamma = [0, 1]$ or $[0, \infty]$) and $\mathcal{F}(y) = u(\phi(y))$ is analytic on $[-1, 1]$, then the error of the approximation can be bounded as follows:*

$$e_N \leq \sum_{i=N}^{\infty} \frac{\pi 2^{1-2i-2\alpha}\Gamma(i+2\alpha)M_i^2}{(i+\alpha)\Gamma^2(i+\alpha)\Gamma(i+1)} \tag{34}$$

where $\phi(y) = L\left(\frac{1+y}{1-y}\right)$ or $\phi(y) = 2y - 1$ and $M_i = \max |\mathcal{F}^{(i)}(y)|, y \in (-1, 1)$.

Although this theorem expresses the approximation convergence rate of the collocation method, but we can improve this rate for special case when $\alpha \geq 0$ to supper-linear convergence rate.

Theorem 2 *The convergence rate of shifted and rational Gegenbauer collocation method for $\alpha \geq 0$ is super-linear.*

Proof. First, we show that $\frac{\Gamma(i+2\alpha)}{\Gamma(i+\alpha)} < 2^i \times \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}$ as follow:

$$\begin{aligned}\frac{\Gamma(i+2\alpha)}{\Gamma(i+\alpha)} &= \frac{(i+2\alpha-1)(i+2\alpha-2)\dots 2\alpha\Gamma(2\alpha)}{(i+\alpha-1)(i+\alpha-2)\dots\alpha\Gamma(\alpha)} \\ &= \underbrace{\frac{(i+2\alpha-1)}{(i+\alpha-1)}}_{<2} \times \underbrace{\frac{(i+2\alpha-2)}{(i+\alpha-2)}}_{<2} \times \dots \times \underbrace{\frac{(1+2\alpha)}{(1+\alpha)}}_{<2} \times \underbrace{\frac{2\alpha}{\alpha}}_{\leq 2} \times \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} < 2^i \times \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}.\end{aligned} \tag{35}$$

Now, let consider $M \geq M_i$ for $i \geq N$. So, Eq. (34), can be rewritten as the following:

$$e_N \leq \underbrace{(\pi M^2 2^{1+2\alpha})\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}}_{S_\alpha} \sum_{i=N}^{\infty} \frac{2^{-2i} \times 2^i}{i!\Gamma(i+\alpha+1)} = S_\alpha \sum_{i=N}^{\infty} \frac{2^{-i}}{i!\Gamma(i+\alpha+1)} \leq S_\alpha \sum_{i=N}^{\infty} \frac{2^{-i}}{(i!)^2}. \tag{36}$$

The sequence $a_N = \sum_{i=N}^{\infty} \frac{2^{-i}}{(i!)^2}$ is super-linearly convergent to zero and so, $\{e_N\}$ is also converges to zero super-linearly. □

3. Lane-Emden type singular differential equations

It is common in several non-Newtonian models of fluid mechanics, mathematical physics, and astrophysics to encounter singular initial-value problems in ordinary differential equations [96, 97]. Examples of theories that are modeled using Lane-Emden equations include the theories of the internal structure of stars, galaxy clusters, the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules, and theories of thermionic currents. In essence, the polytropic theory of stars derives from thermodynamic considerations, which address the problem of energy transfer through the movement of material across various layers of the star. Numerous studies have concentrated on these equations, which are some of the fundamental ones in the theory of star structure [42, 43, 44, 47, 48, 49, 50, 55, 56, 57, 58, 59, 64, 65, 67, 68, 69, 98].

Lane-Emden equations have two the following forms:

$$1. \begin{cases} y''(x) + \frac{k_1}{a(x)}y'(x) + \frac{k_2}{b(x)}y(x) + f(x, y) = 0, & 0 < x \leq 1, \\ y(0) = A, \ y'(0) = B, \end{cases} \tag{37}$$

where A, B, k_1 and k_2 are real constants, $a(x)$ and $b(x)$ are continuous and maybe $a(0) = 0, b(0) = 0$ and $f(x, y)$ is a continuous real valued function [99].

$$2. \begin{cases} y''(x) + \frac{2}{x}y'(x) + f(y) = 0, & x \geq 0, \\ y(0) = A, \ y'(0) = B, \end{cases} \tag{38}$$

where A and B are real constants, $f(y)$ is a given function and x is defined on semi-infinite domain.

We will have the standard Lane-Emden equation with $f(y) = y^m$ that values of m lie in the interval $[0, 5]$:

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0 \quad x \geq 0, \tag{39}$$

which may also be written as

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^m(x), \tag{40}$$

with the same boundary conditions which were mentioned previously [51].

4. Solving Lane-Emden equations

4.1. Lane-Emden problems defined on finite domain

In this part, we apply collocation method via the shifted Gegenbauer-Gauss points to approximate the Lane-Emden equations that are determined in $[0, 1]$. So we should expand $y(x)$ in equations as given in the following:

$$I_N y(x) = \sum_{k=0}^{N} c_k SG_k^{\alpha}(x), \tag{41}$$

where $SG_k^{\alpha}(x)$ are shifted Gegenbauer (SG) polynomials that can be obtained by $\psi(x) = \frac{1}{2}(x + 1)$ mapping. After that, we construct the residual function $(Res(x))$ by substituting $y(x)$ by the above expansion in the equation and its boundary conditions. By equalizing $Res(x)$ to zero at $N - 1$ shifted Gegenbauer-Gauss points and two boundary conditions, we have $N + 1$ equations that generate a set of $N + 1$ nonlinear equations that can be solved by Newton method for the unknown coefficients c_k.

The advantage of solving problems with Gegenbauer polynomials as orthogonal functions is the existence of α parameter which can be varied to obtain a better solution.

In the demonstrated examples of this section, the comparison of values of $y(x)$ obtained by the presented method and exact values will be presented.

4.1.1. **Example 1** Consider the following nonlinear Lane-Emden type equation:

$$\begin{cases} y''(x) + \frac{2}{x}y'(x) + y(x) - (x^3 + x^2 + 12x + 6) = 0, & 0 < x \leq 1, \\ y(0) = 0, \ y'(0) = 0. \end{cases} \tag{42}$$

It is easy to see that the exact solution is $y(x) = x^2 + x^3$.

This equation has been solved by [56, 68, 100, 101] with linearization, VIM, HPM and TSADM methods, respectively.

Solution: First we make the following equations and then we solve the system with $N + 1$ equations to get c_k.

$$\begin{aligned} &Res(x) = \frac{d^2}{dx^2}I_N y(x) + \frac{2}{x}\frac{d}{dx}I_N y(x) + I_N y(x) - (x^3 + x^2 + 12x + 6), \\ &Res(x_j) = 0, \quad j = 1, 2, ..., N - 1, \\ &I_N y(0) = 0, \\ &\frac{d}{dx}I_N y(x)\Big|_{x=0} = 0, \end{aligned} \tag{43}$$

where x_j are shifted Gegenbauer-Gauss points with $\alpha > -\frac{1}{2}$.

The answer for $y(x)$ with $N \geq 3$ and all $\alpha > -\frac{1}{2}$ is:

$$-0.5 + 1.75x + 0.625(2x - 1)^2 + 0.125(2x - 1)^3 = x^2 + x^3, \tag{44}$$

whose simplified form is equal to the exact solution.

4.1.2. **Example 2** Consider the following problem:

$$\begin{cases} y''(x) + \frac{8}{x}y'(x) + xy(x) - (x^5 + x^4 - 44x^2 + 30x) = 0, & 0 < x \leq 1, \\ y(0) = 0, \ y'(0) = 0. \end{cases} \tag{45}$$

It is easy to see that the exact solution is $y(x) = x^4 - x^3$.

This type of equation has been solved by [56, 64, 65, 101] with linearization, HPM, HAM and TSADM methods respectively.

Solution: The same as example 1, we solve the system of this problem to obtain c_k and the obtained answer for $y(x)$ with $N \geq 4$ and all $\alpha > -\frac{1}{2}$ is:

$$0.0625 - 0.25x + 0.125(2x - 1)^3 + 0.0625(2x - 1)^4 = x^4 - x^3, \tag{46}$$

whose simplified form is equal to the exact solution.

4.1.3. **Example 3** Consider the following singular initial value problem:

$$\begin{cases} y''(x) + \frac{2}{x}y'(x) - 6y(x) - 4x^2y(x) = 0, & 0 < x \leq 1, \\ y(0) = 1, \ y'(0) = 0. \end{cases} \tag{47}$$

The exact solution is $y(x) = e^{x^2}$.

This type of equation has been solved by [56, 68, 100] with linearization, VIM and HPM methods respectively.

Solution: We solve this example with some cases of α, i.e. $\alpha = -0.45, 0, 0.5, 1, 1.5$ and 2 and then, for choosing a more acceptable value for α parameter, compare their results in Table 1 by measuring the $\|Res\|^2$ as:

$$\|Res\|^2 = \int_0^1 Res^2(x)\ dx.$$

The validity of the method is based on the assumption that it converges by increasing the number of Gauss points. Taking this table into account, the proposed method leads to more accurate solutions with high convergence by increasing N. As can be seen, $\alpha = -0.45$ has more accurate result among the reported values. So, we show the logarithmic graph of absolute coefficients $|c_k|$ of SG functions in the approximate solutions of the problem, with $\alpha = -0.45$ and $N = 20$ in Fig. 1.

4.1.4. **Example 4** Consider the Lane-Emden type equation as follows:

$$\begin{cases} y''(x) + \frac{2}{x}y'(x) + 8e^{y(x)} + 4e^{\frac{y(x)}{2}} = 0, & 0 < x \leq 1, \\ y(0) = 0, \ y'(0) = 0. \end{cases} \tag{48}$$

The exact solution of this problem is $-2ln(1 + x^2)$.

This type of equation has been solved by [68, 100] with VIM and HPM methods respectively.

Solution: Similar to the previous example, we solve this equation with various values of α parameter to approximate $y(x)$ and in Table 2, we show the convergence rate and accuracy of this method for these values by presenting $\|Res\|^2$, which was already defined. In this example again $\alpha = -0.45$ has better result and we plot the logarithmic graph of absolute coefficients $|c_k|$ of SG functions by this parameter with $N = 20$ in Fig. 2.

4.2. Lane-Emden problems defined on semi-infinite domain

This section applies the P_N operator to the $y(x)$ function under Eq. (31). The residual function is then created by replacing y(x) with $P_N y(x)$ in the model equation (38). We may determine the coefficients a_k by equalizing $Res(x)$ to 0 at rational Gegenbauer-Gauss points ($x_j, j = 1, 2, ..., N-1$) and two boundary conditions. We see that a set of $N+1$ nonlinear equations resulting from these $N+1$ equations can be solved using a well-known technique like the Newton method.

Two parameters in the rational Gegenbauer collocation method (RGC) can be changed to produce better solutions to issues. These variables are L and $alpha$.

The following examples are defined on the semi-infinite domain. We solve them and compare the presented work with some well-known results afterward.

4.2.1. **Example 1** Consider the standard Lane-Emden equation:

$$\begin{cases} y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, & x \geq 0, \\ y(0) = 1, \ y'(0) = 0. \end{cases} \tag{49}$$

The values of m which are physically interesting, lie in the interval $[0, 5]$. Exact solutions for Eq. (49) are known only for the values $m = 0, 1$ and 5. For other values of m, the Lane-Emden equation is to be integrated numerically. Horedt [102] found exact values for the first zeros and values of $y(x)$ for this equation by different value of m.

Solution: By solving the following equations, we approximate $y(x)$ and proceed to obtain a_k.

$$\begin{aligned} &Res(x) = \frac{d^2}{dx^2}P_N y(x) + \frac{2}{x}\frac{d}{dx}P_N y(x) + P_N y^m(x), \\ &Res(x_j) = 0, \quad j = 1, 2, ..., N-1, \\ &P_N y(0) = 1, \\ &\frac{d}{dx}P_N y(x)\Big|_{x=0} = 0, \end{aligned} \tag{50}$$

where x_j are rational Gegenbauer-Gauss points.

We first solve this equation by rational Gegenbauer function with some cases of α and varied L to obtain the first zeros of the equations equal to exact values which were obtained by Horedt [102] for $m = 2, 3$ and 4 and report them in Table 3. Then, to compare the results which are related to each α, we present the values of $y(x)$ for $m = 2, 3$ and 4 that are obtained by presented method and those obtained by Horedt [102] in Tables 4, 5 and 6, respectively. In addition, we report the $\|Res\|^2$ of each of them as:

$$\|Res\|^2 = \int_0^\infty Res^2(x)\ dx,$$

in these tables. According to these tables, we can find that $\alpha = 0.5, 0.5$ and 1 are more acceptable value of α parameter among the reported values for $m = 2, 3$ and 4, respectively.

Moreover, in Tables 7 and 8, $\|Res\|^2$ that obtained by solving Lane-Emden equation for $m = 3$ and $m = 4$ with $N = 10$ and $N = 20$ are reported to show the convergence and accuracy of our method. In these tables, we also present the values of L parameter for each α. Furthermore, we plot the logarithmic graphs of absolute coefficients $|a_k|$ of RG functions in the approximate solutions of the standard Lane-Emden equation for $m = 3$ and $m = 4$ with $\alpha = 0.5$ and 1 by $N = 20$ in Figs. 3 and 4, respectively. These graphs illustrate that the method has an appropriate convergence rate.

The resulting graph of the standard Lane-Emden equation for $m = 2, 3$ and 4 is shown in Fig. 5.

4.2.2. **Example 2** Consider the isothermal gas spheres equation:

$$\begin{cases} y''(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0, \quad x \geq 0, \\ y(0) = 0,\ y'(0) = 0. \end{cases} \tag{51}$$

A series solution has been obtained by Wazwaz [47], Liao [49], Singh et al. [66] and Ramos [58] by using ADM, HAM, MHAM and series expansion respectively.

This equation has been solved by [47, 49, 51, 58, 64, 65, 66, 98] and we compare our solution with results presented by Wazwaz [47].

Solution: The same as previous example, we solve this equation with some cases of α parameter and present values of $y(x)$ obtained by RGC method with $N = 20$ and $L = 1$ and compare them with the results reported by Wazwaz [47] in Table 9. In Fig. 6, we present the graph of $y(x)$ that obtained by solving this example with proposed method.

4.2.3. **Example 3** Consider the following Lane-Emden equation:

$$\begin{cases} y''(x) + \frac{2}{x}y'(x) + sin(y(x)) = 0, \quad x \geq 0, \\ y(0) = 1,\ y'(0) = 0. \end{cases} \tag{52}$$

This equation has been solved by [47, 51] by ADM (Adomian Decomposition Method) and HFC (Hermit Functions Collocation) methods. We compare our results with those presented by Wazwaz [47].

Solution: The value of $y(x)$ obtained by solving this equation with different α, $N = 20$ and $L = 1$ and results given by Wazwaz [47] are shown in Table 10. In addition, the resulting graph of this example is shown in Fig. 7.

4.2.4. **Example 4** Consider the following nonlinear initial value Lane-Emden type equation:

$$\begin{cases} y''(x) + \frac{2}{x}y'(x) + sinh(y(x)) = 0, \quad x \geq 0, \\ y(0) = 1,\ y'(0) = 0. \end{cases} \tag{53}$$

This equation has been solved by [47, 51] with ADM and HFC methods.

Solution: In order to compare the presented method with those obtained by Wazwaz [47], the value of $y(x)$ of this problem for different values of α, $N = 20$ and $L = 1$ are presented in Table 11. Also, the graph of the approximation of $y(x)$ is shown in Fig. 8.

5. Conclusions

In order to solve nonlinear singular initial value problems, particularly those involving generalized Lane-Emden type equations, we used the collocation approach. This technique is simple to use and produces the needed accuracy. The selection of the basis functions for the collocation approach is a crucial consideration. By assuming that the truncation N is large enough, any solution can be represented to arbitrarily high accuracy thanks to the basis functions' significant qualities of fast computation, quick convergence, and completeness. In this paper, the basis functions were rational Gegenbauer functions and shifted Gegenbauer polynomials. Some Lane-Emden type equations in the interval $[0, 1]$ and semi-infinite domain have been successfully solved using the collocation approach by these functions. The advantage of solving problems with these Gegenbauer functions as orthogonal functions is the existence of the parameter α that can be varied to reach better solutions. We proved the super-linear convergence rate of our method, theoretically and got help from $\|Res\|^2$ to show the convergence rate and accuracy of the method, numerically.

Table 1. $\|Res\|^2$ that obtained by presented method with various values of α in solving example 3 in the finite domain problems part.

α	$N=10$	$N=15$	$N=20$
−0.45	3.7184e−07	1.6355e−15	5.0153e−25
0	5.0370e−07	2.9069e−15	1.1080e−24
0.5	6.5537e−07	4.9360e−15	2.3380e−24
1	8.0744e−07	7.6951e−15	4.4606e−24
1.5	9.5636e−07	1.1240e−14	7.8555e−24
2	1.0998e−06	1.5600e−14	1.2926e−23

Table 2. $\|Res\|^2$ that obtained by presented method with various values of α in solving example 4 in the finite domain problems part.

α	$N=10$	$N=15$	$N=20$
−0.45	6.8440e−06	2.8626e−11	3.4619e−12
0	9.7548e−06	3.8604e−11	7.6664e−12
0.5	1.3303e−05	8.6230e−11	8.4983e−12
1	1.7034e−05	1.8684e−10	9.4589e−12
1.5	2.0817e−05	2.5247e−10	1.8009e−11
2	2.4577e−05	2.5772e−10	1.8640e−11

Table 3. Comparison the first zero of standard Lane-Emden equation obtained by RGC method and exact numerical values [102].

m	N	RGC method	Exact value
2	10	4.35287460	4.35287460
3	20	6.89684862	6.89684862
4	20	14.9715463	14.9715463

Table 4. Comparison of $y(x)$ for standard Lane-Emden equation with $m=2$, between RGC (with different α) method when $N=10$ and exact values given by Horedt[102].

x	$\alpha=-0.45$	$\alpha=0$	$\alpha=0.5$	$\alpha=1$	$\alpha=1.5$	$\alpha=2$	Exact value
	$L=0.6278$	$L=0.6700$	$L=0.7122$	$L=0.7488$	$L=0.7833$	$L=0.8176$	[102]
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.9982747	0.9983050	0.9983418	0.9983630	0.9983613	0.9983356	0.9983350
0.5	0.9587730	0.9590561	0.9592533	0.9593512	0.9593880	0.9593829	0.9593527
1.0	0.8503391	0.8496522	0.8491799	0.8489235	0.8487759	0.8486836	0.8486541
3.0	0.2390090	0.2401446	0.2406964	0.2410017	0.2411283	0.2411569	0.2418241
4.0	0.0480526	0.0487107	0.0483886	0.0482222	0.0480321	0.0478330	0.0488401
4.3	0.0061012	0.0069678	0.0067036	0.0066653	0.0066246	0.0065833	0.0068109
4.35	0.0006226	0.0005968	0.0003597	0.0003575	0.0003552	0.0003529	0.0003660
$\|Res\|^2$	2.40e−03	1.30e−03	7.82e−04	8.52e−04	1.55e−03	2.99e−03	

Table 5. Comparison of $y(x)$ for standard Lane-Emden equation with $m=3$, between RGC (with different α) method when $N=20$ and exact values given by Horedt [102].

x	$\alpha=-0.45$	$\alpha=0$	$\alpha=0.5$	$\alpha=1$	$\alpha=1.5$	$\alpha=2$	Exact value
	$L=0.9902$	$L=1.5636$	$L=1.8550$	$L=1.9441$	$L=2.0353$	$L=2.1264$	[102]
0.0	1.0000126	1.0000000	1.0000126	1.0000000	1.0000126	1.0000000	1.0000000
0.1	0.9983351	0.9983362	0.9983360	0.9983359	0.9983357	0.9983358	0.9983358
0.5	0.9598286	0.9598429	0.9598382	0.9598384	0.9598388	0.9598394	0.9598391
1.0	0.8550313	0.8550683	0.8550557	0.8550562	0.8550569	0.8550577	0.8550576
5.0	0.1112628	0.1107056	0.1108296	0.1108270	0.1108252	0.1108237	0.1108198
6.0	0.0444607	0.0434548	0.0437481	0.0437440	0.0437419	0.0437407	0.0437380
6.8	0.0042795	0.0041250	0.0041703	0.0041693	0.0041686	0.0041683	0.0041678
6.896	0.0000370	0.0000356	0.0000360	0.0000360	0.0000360	0.0000360	0.0000360
$\|Res\|^2$	1.63e−05	4.86e−06	7.99e−08	1.16e−07	3.94e−07	1.50e−06	

Table 6. Comparison of $y(x)$ for standard Lane-Emden equation with $m = 4$, between RGC (with different α) method when $N = 20$ and exact values given by Horedt [102].

x	$\alpha = -0.45$ $L = 1.2740$	$\alpha = 0$ $L = 1.3971$	$\alpha = 0.5$ $L = 1.5280$	$\alpha = 1$ $L = 1.6535$	$\alpha = 1.5$ $L = 1.7746$	$\alpha = 2$ $L = 1.8917$	Exact value [102]
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.9983366	0.9983366	0.9983367	0.9983367	0.9983367	0.9983367	0.9983367
0.2	0.9933864	0.9933863	0.9933862	0.9933862	0.9933862	0.9933862	0.9933862
0.5	0.9603112	0.9603111	0.9603109	0.9603109	0.9603109	0.9603109	0.9603109
1.0	0.8608145	0.8608144	0.8608140	0.8608138	0.8608138	0.8608138	0.8608138
5.0	0.2358982	0.2359128	0.2359192	0.2359217	0.2359225	0.2359228	0.2359227
10.0	0.0596108	0.0596517	0.0596659	0.0596706	0.0596723	0.0596729	0.0596727
14.0	0.0082593	0.0082960	0.0083125	0.0083202	0.0083242	0.0083265	0.0083305
14.9	0.0005705	0.0005735	0.0005749	0.0005755	0.0005758	0.0005760	0.0005764
$\|Res\|^2$	5.05e−08	7.49e−09	1.57e−09	7.95e−10	1.22e−09	2.52e−09	

Table 7. $\|Res\|^2$ that obtained by presented method with various values of α in solving standard Lane-Emden equation with $m = 3$ and different N.

	$N = 10$		$N = 20$	
α	L	$\|Res\|^2$	L	$\|Res\|^2$
−0.45	1.1161	8.41e−05	0.9902	1.63e−05
0	1.1658	4.65e−05	1.5636	4.86e−06
0.5	1.2024	3.38e−05	1.8550	7.99e−08
1	1.2277	4.80e−05	1.9441	1.16e−07
1.5	1.2475	1.11e−04	2.0353	3.94e−07
2	1.2666	2.60e−04	2.1264	1.50e−06

Table 8. $\|Res\|^2$ that obtained by presented method with various values of α in solving standard Lane-Emden equation with $m = 4$ and different N.

	$N = 10$		$N = 20$	
α	L	$\|Res\|^2$	L	$\|Res\|^2$
−0.45	1.5990	5.13e−06	1.2740	5.05e−08
0	1.6162	3.84e−06	1.3971	7.49e−09
0.5	1.6585	3.36e−06	1.5280	1.57e−09
1	1.8556	6.55e−07	1.6535	7.95e−10
1.5	2.0782	2.32e−06	1.7746	1.22e−09
2	2.2776	2.40e−05	1.8917	2.52e−09

Table 9. Comparison of $y(x)$ for example 2 in the semi-infinite domain problems part between RGC method when $N = 20$ and $L = 1$ and results given by Wazwaz [47].

x	$\alpha = -0.45$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	[47]
0.0	0.0000002730	0.0000002660	0.0000001100	0.0000000550	0.0000000550	0.0000000360	0.0000000000
0.1	−0.0016677956	−0.0016665980	−0.0016660715	−0.0016660157	−0.0016658577	−0.0016656584	−0.0016658339
0.2	−0.0066547655	−0.0066534760	−0.0066528258	−0.0066530090	−0.0066530782	−0.0066530558	−0.0066533671
0.5	−0.0411666095	−0.0411588736	−0.0411559131	−0.0411549620	−0.0411544204	−0.0411540527	−0.0411539568
1.0	−0.1588571140	−0.1588398270	−0.1588330211	−0.1588304094	−0.1588290990	−0.1588283482	−0.1588273537
1.5	−0.3379983541	−0.3380129425	−0.3380173539	−0.3380190421	−0.3380195160	−0.3380195473	−0.3380131103
2.0	−0.5598857992	−0.5598457955	−0.5598307392	−0.5598254313	−0.5598234401	−0.5598226916	−0.5599626601
2.5	−0.8062215633	−0.8062831163	−0.8063128797	−0.8063271711	−0.8063341436	−0.8063375927	−0.8100196713

Table 10. Comparison of $y(x)$ for example 3 in the semi-infinite domain problems part between RGC method when $N = 20$ and $L = 1$ and results given by Wazwaz [47].

x	$\alpha = -0.45$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	[47]
0.0	1.0000001106	1.0000000300	0.9999999600	1.0000000370	0.9999999440	0.9999999750	1.0000000000
0.1	0.9985268545	0.9985337971	0.9985715030	0.9986027200	0.9986045423	0.9985508146	0.9985979358
0.2	0.9942824626	0.9943917179	0.9944162806	0.9943890391	0.9943543156	0.9943042582	0.9943962733
0.5	0.9642098388	0.9646863473	0.9650262090	0.9651958893	0.9652520039	0.9652180589	0.9651777886
1.0	0.8611661670	0.8624107858	0.8632684148	0.8637138261	0.8638961208	0.8639045113	0.8636811027
1.5	0.7068170726	0.7056937395	0.7051493583	0.7050150012	0.7050449348	0.7050825363	0.7050419247
2.0	0.5006486878	0.5039795671	.50586930513	0.5065212311	0.5065632398	0.5063863111	0.5063720330

Table 11. Comparison of $y(x)$ for example 4 in the semi-infinite domain problems part between RGC method when $N = 20$ and $L = 1$ and results given by Wazwaz [47].

x	$\alpha = -0.45$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	[47]
0.0	1.0000003320	0.9999997830	1.0000000320	1.0000000040	0.9999999640	0.9999998790	1.0000000000
0.1	0.9979693899	0.9979663470	0.9979974487	0.9980310259	0.9980447243	0.9980152317	0.9980428414
0.2	0.9920720162	0.9921841621	0.9922239458	0.9922076806	0.9921771845	0.9921359409	0.9921894348
0.5	0.9509655503	0.9513798283	0.9517081280	0.9518956384	0.9519802170	0.9519841324	0.9519611019
1.0	0.8156715868	0.8167543431	0.8175735172	0.8180511293	0.8182905969	0.8183685057	0.8182516669
1.5	0.6272322965	0.6261933528	0.6256220407	0.6254314378	0.6254210034	0.6254553927	0.6258916077
2.0	0.4007310194	0.4037527164	0.4056767515	0.4064909251	0.4066821960	0.4066083001	0.4136691039

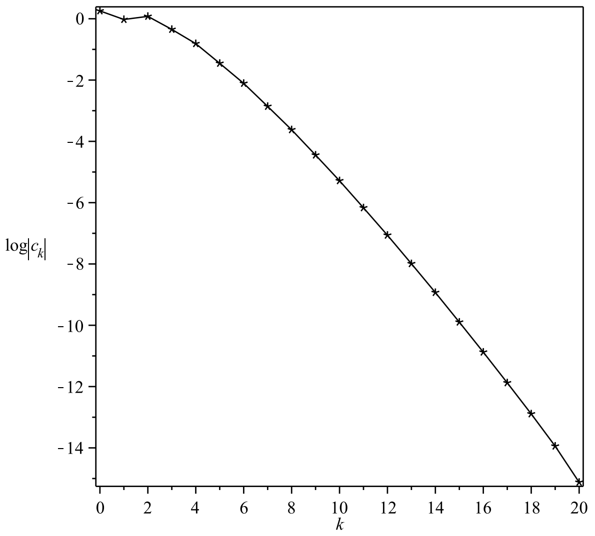

Figure 1. Logarithmic graph of absolute coefficients $|c_k|$ of the shifted Gegenbauer functions for approximating $y(x)$ of example 3 in the finite domain problems part.

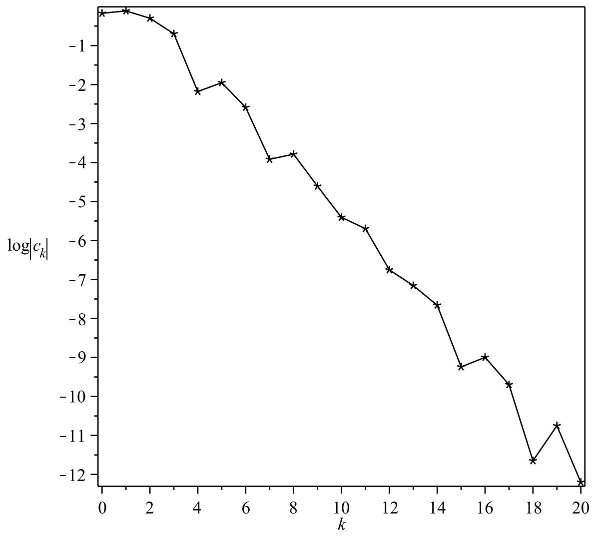

Figure 2. Logarithmic graph of absolute coefficients $|c_k|$ of the shifted Gegenbauer functions for approximating $y(x)$ of example 4 in the finite domain problems part.

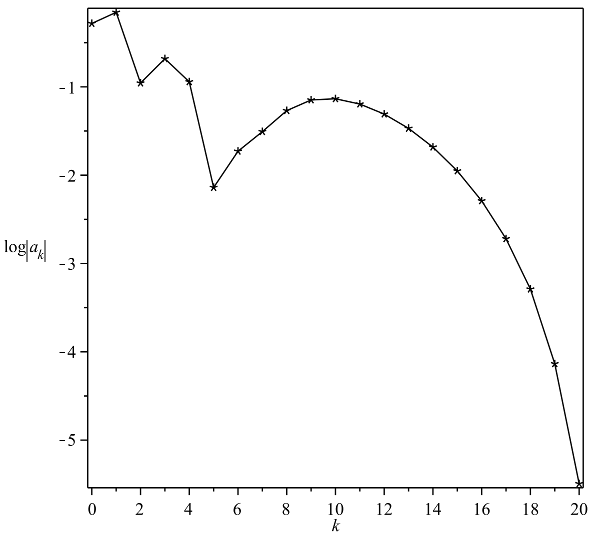

Figure 3. Logarithmic graph of absolute coefficients $|a_k|$ of the rational Gegenbauer functions for approximating $y(x)$ of the standard Lane-Emden equation with $m = 3$.

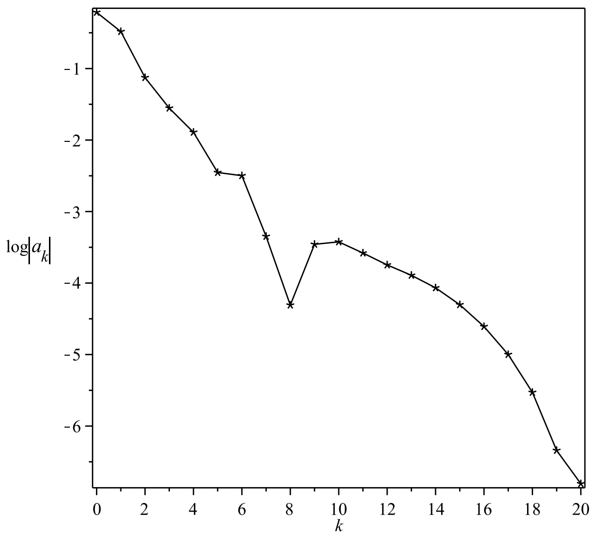

Figure 4. Logarithmic graph of absolute coefficients $|a_k|$ of the rational Gegenbauer functions for approximating $y(x)$ of the standard Lane-Emden equation with $m = 4$.

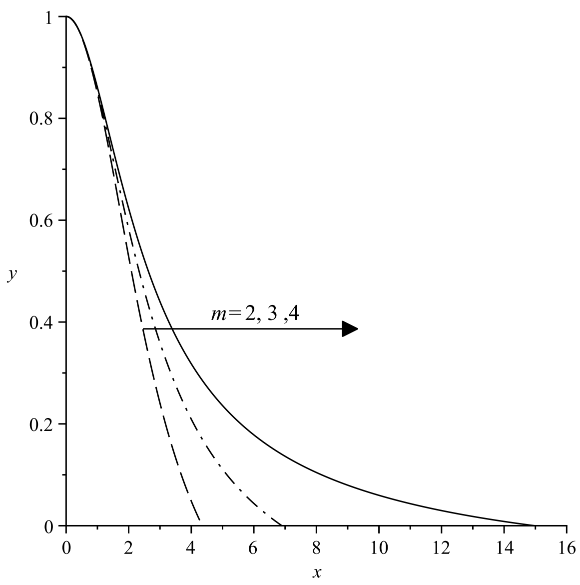

Figure 5. Graph of standard Lane-Emden equation for $m = 2, 3$ and 4.

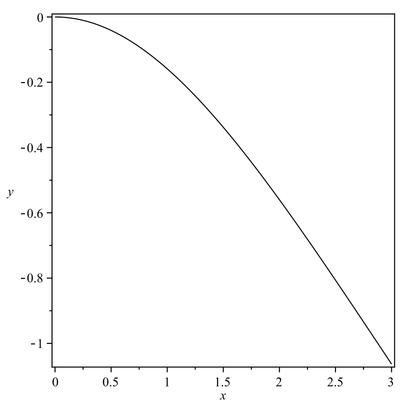

Figure 6. Graph of the approximation of $y(x)$ of example 2 in the semi-infinite domain problems part.

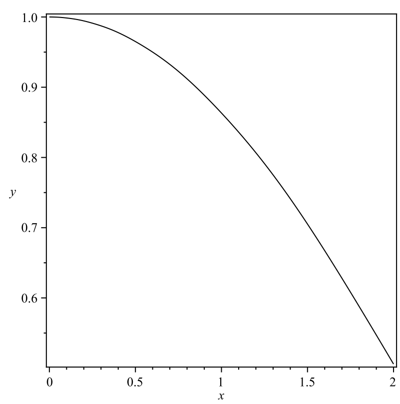

Figure 7. Graph of the approximation of $y(x)$ of example 3 in the semi-infinite domain problems part.

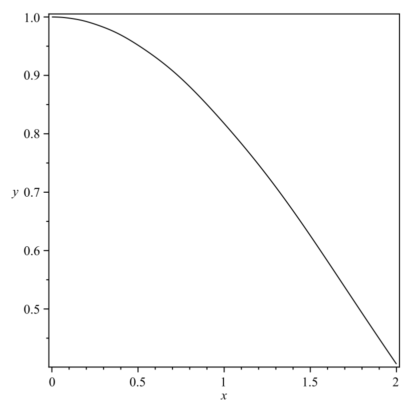

Figure 8. Graph of the approximation of $y(x)$ of example 4 in the semi-infinite domain problems part.

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