A hybrid numerical method based on the generalized pseudospectral method for solving nonlinear differential equations

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In this paper, a hybrid numerical method using generalized pseudospectral and Newton-Kantorovich quasilinearization methods is presented to solve nonlinear differential equations. Initially, generalized Lagrange functions as basic functions are introduced and then derivative operational matrices for these functions are presented. Then using these new functions, the generalized pseudospectral method is constructed as a numerical method. Finally, this method and the Newton-Kantorovich quasilinearization method are combined to produce an efficient method. Because of the use of derivative operating matrices and the conversion of any nonlinear differential equation into sequences of linear differential equations, the implementation of this method does not require mathematically to calculate the derivative and the computational costs are also reduced. To illustrate the efficiency, accuracy, and convergence of the method, the proposed method is implemented on two famous equations and the results are compared with other methods. Copyright © 2022 Shahid Beheshti University.

Keywords: Generalized pseudospectral method; Newton-Kantorovich quasilinearization method; Generalized Lagrange functions; Derivative operational matrix.

Table 1. Symbols

symbol	explanation	symbol	explanation	symbol	explanation
n	Index of repetitions	N	The number of collocation points	$y_0(x)$	Initial function to start iterations
$L_j^\phi(x)$	Generalized Lagrange function	$u_N(x)$	Approximate solution of problem	$D^{(m)}$	The derivative operational matrix of m orders
L	The shape parameter	$Res(x)$	Residual error function	$[a, b]$	The definition interval of equation
λ	Reaction rate parameter	δ	Activation energy parameter	β	Heat evolution parameter

1. Introduction

Many practical and engineering problems are modelled using nonlinear differential equations and solving these types of equations with analytical methods is not possible or efficient, for this, numerical methods are often used to solve them. In this regard, in this research, a new numerical method, called the generalized pseudospectral method [1], was combined with the quasi-linearization method, and then by using it, the solution of two important nonlinear differential equations was investigated.

1.1. Newton-Kantorovich quasi-linearization method

Bellman and Kalaba introduced the Newton-Kantorovich quasi-linearization method based on the Newton-Raphson method for solving nonlinear differential equations of n-orders [2, 3]. In this method, the desired nonlinear differential equation is converted into a sequence of linear differential equations, so that this sequence converges uniformly to the solution of the

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desired nonlinear differential equation [4]. Several researchers have used this method and its generalizations in their papers and researches [5, 6, 7, 8, 9, 10, 11, 12].

In this research, a new pseudospectral method is used to solve the linear differential equations generated in each iteration of the Newton-Kantorovich method.

Consider the nonlinear differential equation of the second order in the below form on the interval $[a, b]$, where a and b can be infinite numbers:

$$\frac{d^2 y}{dx^2} = F\left(y'(x), y(x), x\right), \tag{1}$$

with the corresponding initial or boundary conditions, such as $y(0) = A$ and $y'(0) = B$, that A and B are real parameters.

According to the Newton-Kantorovich quasi-linearization method, the solution of equation (1) is the same as the solution of the following linear differential equation in the $(n+1)$-th iteration:

$$\frac{d^2 y_{n+1}}{dx^2} = F\left(y'_n, y_n, x\right) + (y_{n+1} - y_n) F_y\left(y'_n, y_n, x\right) + \left(y'_{n+1} - y'_n\right) F_{y'}\left(y'_n, y_n, x\right), \tag{2}$$

with the corresponding initial or boundary conditions:

$$y_{n+1}(0) = A, \qquad\qquad y'_{n+1}(0) = B, \tag{3}$$

where the functions of $F_y = \frac{\partial F}{\partial y}$ and $F_{y'} = \frac{\partial F}{\partial y'}$ are the functional derivative of $F(y', y, x)$ and the prime notation indicates a typical derivative to the variable x and $n=0, 1, 2, \ldots$.

In the quasi-linearization method, the start of repetitions requires an initial value or initial guess of $y_0(x)$, which is usually selected based on boundary or initial conditions and physical conditions of the equation. Mandelzweig and Tabakin [4] have proven that if this initial guess satisfies to at least one of the initial or boundary conditions of the equation, this method would be convergence.

1.2. Generalized pseudospectral method

Pseudospectral (interpolation) methods play an important role in solving differential equations. In such methods, some interpolation points ($\{x_i\}_{i=0}^N$) are used to construct interpolation functions (such as Lagrange functions). Based on the structure of collocation methods, several collocation points ($\{y_i\}_{i=0}^N$) are used to zero the residual error function. In general, these two types of interpolation functions and collocation points can be different, but in most cases, these points are considered the same to reduce computational costs.

1.2.1. Generalized Lagrange functions Before introducing the generalized pseudospectral method [1] as a generalization of the Lagrange method, there is a need to define the generalized Lagrange functions, which are discussed in this section.

Definition 1: Generalized Lagrange functions, as new basis functions, are defined as follows [1]:

$$L_j^\phi(x) = \prod_{\substack{i=0 \\ i \neq j}}^N \frac{\phi(x) - \phi(x_i)}{\phi(x_j) - \phi(x_i)}, \qquad\qquad j = 0, \ldots, N, \tag{4}$$

where the interpolation points $x_0 < x_1 < \cdots < x_N$ are real and arbitrary points in the interval $[a, b]$, $\phi(x)$ is an arbitrary and sufficiently smooth and differentiable function on the interval (a, b) so that $\phi'(x_i) \neq 0$ and $\phi(x_j) \neq \phi(x_i)$ for every $i \neq j$.

By defining $w^\phi(x) = \prod_{i=0}^N \phi(x) - \phi(x_i)$, the generalized Lagrange function can be rewritten as follows:

$$L_j^\phi(x) = \gamma_j \frac{w^\phi(x)}{\phi(x) - \phi(x_j)}, \tag{5}$$

Where $\gamma_j = \frac{\phi'(x_j)}{\left(w^\phi(x)\right)'\big|_{x=x_j}}$.

Remark 1: The values of a and b are selected based on the definition interval of the $\phi(x)$ function, and these values can also have infinite values.

Some properties of generalized Lagrange functions are:

1. These functions have the property of Kronecker delta in the interpolation points, that is, $L_j^\phi(x_i) = \delta_{ij}$, which reduces the cost of calculations.
2. If the function $\phi(x)$ is bounded, the functions $L_j^\phi(x)$ are also bounded, which is very useful in problems for infinite and semi-infinite intervals.
3. It has the property of $\sum_{j=0}^N L_j^\phi(x) = 1$.
4. There is no condition for the function $\phi(x)$ to be invertible. This feature can be very useful for equations that its answer structure is irregular and intermittent.
5. Different choices of the function $\phi(x)$ provide the possibility of generating different basic functions.

1.2.2. Generalized pseudospectral method With a suitable choice of the function $\phi(x)$, the solution of a differential equation can be approximated using generalized Lagrange functions as follows:

$$u(x) \approx u_N(x) = \sum_{j=0}^{N} u_N(x_j) \ L_j^\phi(x). \tag{6}$$

Now, to calculate the m-th derivative of this function at point x_k, it is done as follows:

$$\frac{d^m}{dx^m} u_N(x)\Big|_{x=x_k} = \sum_{j=0}^{N} u_N(x_j) \ \frac{d^m}{dx^m} L_j^\phi(x)\Big|_{x=x_k} = \sum_{j=0}^{N} u_N(x_j) \, \boldsymbol{D}_{kj}^{(m)},$$

or in the form of a matrix

$$U_N^{(m)} = \boldsymbol{D}^{(m)} U_N,$$

where $U_N = [u_N(x_0), u_N(x_1), \cdots, u_N(x_N)]^T$, $U_N^{(m)} = \left[u_N^{(m)}(x_0), u_N^{(m)}(x_1), \cdots, u_N^{(m)}(x_N)\right]^T$ and $\boldsymbol{D}^{(m)} = \left[\boldsymbol{D}_{kj}^{(m)}\right]$ that its elements are in the form of $\boldsymbol{D}_{kj}^{(m)} = \frac{d^m}{dx^m} L_j^\phi(x)\Big|_{x=x_k}$.

The matrix $\boldsymbol{D}^{(m)}$, $m \in \mathbb{N}$, is called the derivative operational matrix, where its elements are obtained using the following two theorems:

Theorem 1: According to equations (5) and (6) and assuming that $\boldsymbol{D}^{(1)} = \left[\boldsymbol{D}_{kj}^{(1)}\right]$ is the operational matrix of the first-order derivative of the generalized Lagrange functions at point x_k, Then the values of $\boldsymbol{D}_{kj}^{(1)}$ for $k, j = 0, \cdots, N$ are calculated as follows:

$$\boldsymbol{D}_{kj}^{(1)} = \begin{cases} \dfrac{\left(w^\phi(x)\right)'\big|_{x=x_k}}{\left(w^\phi(x)\right)'\big|_{x=x_j}} \dfrac{\phi'(x_j)}{\phi(x_k)-\phi(x_j)} & k \neq j, \\[4mm] \dfrac{\left(w^\phi(x)\right)''\big|_{x=x_j}}{2\left(w^\phi(x)\right)'\big|_{x=x_j}} - \dfrac{\phi''(x_j)}{2\phi'(x_j)} & k = j, \end{cases}$$

Proof: See Ref. [1].

Theorem 2: According to equations (5) and (6), Theorem 1 and assuming that $\boldsymbol{D}^{(m)}$ is the operational matrix of the m-th derivative of the generalized Lagrange functions at point x_k, then for each $m = 2$, the matrix $\boldsymbol{D}^{(m)}$ is calculated as follows:

$$\boldsymbol{D}^{(2)} = \left(\boldsymbol{P}\boldsymbol{D}^{(1)} + \boldsymbol{P}^{(1)}\right) \boldsymbol{P}^{-1} \boldsymbol{D}^{(1)},$$

$$\boldsymbol{D}^{(3)} = \left(\boldsymbol{P}\boldsymbol{D}^{(2)} + 2\boldsymbol{P}^{(1)}\boldsymbol{D}^{(1)} + \boldsymbol{P}^{(2)}\right) \boldsymbol{P}^{-1} \boldsymbol{D}^{(1)},$$

and in general

$$\boldsymbol{D}^{(m)} = \left(\sum_{k=0}^{m-1} \binom{m-1}{k} \boldsymbol{P}^{(k)} \boldsymbol{D}^{(m-1-k)}\right) \boldsymbol{P}^{-1} \boldsymbol{D}^{(1)},$$

where $\boldsymbol{P}^{(k)}$ is a diagonal matrix in the form of $\boldsymbol{P}^{(k)} = Diag\left(\phi^{(k+1)}(x_0), \phi^{(k+1)}(x_1), \cdots, \phi^{(k+1)}(x_N)\right)$ and $\boldsymbol{P}^{(0)} = \boldsymbol{P}$.

Proof: See Ref. [1].

Remark 2:

1. If the function $\phi(x) = x$ is chosen, Theorems 1 and 2 become theorems of ordinary Lagrange functions (See page 65 of Ref. [13]).
2. According to equation (4), we have $\phi'(x_i) \neq 0$ for all i, for this, the condition of invertibility of the matrix of $\boldsymbol{P}$ is guaranteed.
3. According to the properties of the generalized Lagrange functions, it is obvious that $\sum_{j=0}^{N} \boldsymbol{D}_{kj}^{(m)} = 0$ for any k. It means that the sum of the elements of each row in the derivative operational matrix is equal to zero.

Lemma 1: With the assumptions of Theorems 1 and 2, we have:

A) *The rational functions in infinite and semi-infinite intervals*: if $\phi(x) = \frac{x}{\sqrt{x^2+L^2}}$ (in the infinite interval) or $\phi(x) = \frac{x-L}{x+L}$ (in the semi-infinite interval) and the interpolation points $x_i = L \ t_i$, where L is an arbitrary positive parameter and the t_i points are arbitrary according to the desired interval, then the following relation exists between $\boldsymbol{D}_L^{(m)}$ (derivative operational matrix of m-order for arbitrary L) and $\boldsymbol{D}_1^{(m)}$ (derivative operational matrix of m-order for $L = 1$):

$$\boldsymbol{D}_L^{(m)} = \frac{1}{L^m} \boldsymbol{D}_1^{(m)},$$

B) *The shifted functions in the finite interval*: if $\phi(x) = \frac{2}{b-a}\left(x - \frac{b+a}{2}\right)$ and the interpolation points $x_i = (b-a)\,t_i$ and the t_i points are arbitrary according to the desired interval, then the following relation exists between $\boldsymbol{D}_{[a,b]}^{(m)}$ (derivative operational matrix of m-order on the interval $[a, b]$) and $\boldsymbol{D}_{[0,1]}^{(m)}$ (derivative operational matrix of m-order on the interval $[0, 1]$):

$$\boldsymbol{D}_{[a,b]}^{(m)} = \frac{1}{(b-a)^m}\boldsymbol{D}_{[0,1]}^{(m)},$$

C) *Fractional shifted function in finite interval*: If $\phi(x) = 2\left(\frac{x}{\eta}\right)^{\alpha} - 1$ and interpolation points $x_i = \eta\,t_i$ where a and η are two arbitrary positive parameters and the t_i points are arbitrary according to the desired interval. Then the following relationship exists between $\boldsymbol{D}_{[0,\eta]}^{(m)}$ (derivative operational matrix of m-order on the interval $[0, \eta]$) and $\boldsymbol{D}_{[0,1]}^{(m)}$ (derivative operational matrix of m-order on the interval $[0, 1]$):

$$\boldsymbol{D}_{[0,\eta]}^{(m)} = \frac{1}{\eta^m}\boldsymbol{D}_{[0,1]}^{(m)},$$

Proof: **A**) According to $\phi(x) = \frac{x-L}{x+L}$, we have:

$$\phi(x)|_{x=x_k} = \frac{x_k - L}{x_k + L} = \frac{L\,t_k - L}{L\,t_k + L} = \frac{t_k - 1}{t_k + 1},$$

and

$$\phi'(x)|_{x=x_k} = \frac{1}{x_k + L} - \frac{x_k - L}{(x_k + L)^2} = \frac{1}{L}\left(\frac{1}{t_k + 1} - \frac{t_k - 1}{(t_k + 1)^2}\right) = \frac{1}{L}A,$$

where A is a number for the state L=1 and in the same way we have:

$$\phi^{(m)}(x)\Big|_{x=x_k} = \frac{1}{L^m}A, \tag{7}$$

and

$$\left(w^{\phi}(x)\right)^{(m)}\Big|_{x=x_k} = \frac{1}{L^m}A,$$

Now according to Theorem 1, for $k \neq j$ we have:

$$\left(\boldsymbol{D}_{kj}^{(1)}\right)_L = \frac{\left(w^{\phi}(x)\right)'\big|_{x=x_k}}{\left(w^{\phi}(x)\right)'\big|_{x=x_j}}\frac{\phi'(x_j)}{\phi(x_k) - \phi(x_j)} = \frac{\frac{1}{L}A}{\frac{1}{L}A}\frac{\frac{1}{L}A}{\phi(x_k) - \phi(x_j)} = \frac{1}{L}A,$$

And for $k=j$, we have:

$$\left(\boldsymbol{D}_{jj}^{(1)}\right)_L = \frac{\frac{1}{L^2}}{2\frac{1}{L}}A - \frac{\frac{1}{L^2}}{2\frac{1}{L}}A = \frac{1}{L}A,$$

Therefore, for $m = 1$, the relationship is established. Now for $m = 2$, we have:

$$\left(\boldsymbol{D}^{(2)}\right)_L = \left(\boldsymbol{P}_L\left(\boldsymbol{D}^{(1)}\right)_L + \left(\boldsymbol{P}^{(1)}\right)_L\right)\left(\boldsymbol{P}^{-1}\right)_L\left(\boldsymbol{D}^{(1)}\right)_L$$

$$= \left(\frac{1}{L}\boldsymbol{P}_1\frac{1}{L}\left(\boldsymbol{D}^{(1)}\right)_1 + \frac{1}{L^2}\left(\boldsymbol{P}^{(1)}\right)_1\right)L\left(\boldsymbol{P}^{-1}\right)_1\frac{1}{L}\left(\boldsymbol{D}^{(1)}\right)_1 = \frac{1}{L^2}\left(\boldsymbol{P}_1\left(\boldsymbol{D}^{(1)}\right)_1 + \left(\boldsymbol{P}^{(1)}\right)_1\right)\left(\boldsymbol{P}^{-1}\right)_1\left(\boldsymbol{D}^{(1)}\right)_1 = \frac{1}{L^2}\left(\boldsymbol{D}^{(2)}\right)_1,$$

That is, the relationship is also established for $m = 2$. In the same way and according to the definition of $\boldsymbol{D}^{(m)}$ in Theorem 2 and equation (7), the proof of (A) is complete. Other parts of the lemma can be proved in the same way. ∎
Therefore, by using derivative operational matrices in the unit interval or for $L = 1$, other derivative operational matrices can be easily calculated in any interval or any value of L.

2. Implementation of generalized pseudospectral method

In this section, we implement the generalized pseudospectral method on the nonlinear differential equations (1) and compare the obtained results with other methods.
According to the quasi-linearization method, the solution of equation (1) is the solution of the $(n+1)$-th iteration of the linear differential equation (2), which can be rewritten as follows:

$$y''_{n+1} + q_1(x)\,y'_{n+1} + q_0(x)\,y_{n+1} = f_n(x), \tag{8}$$

where $q_0(x)$, $q_1(x)$ and $f_n(x)$ are known functions and are calculated from the following relations:

$$q_1(x) = -F_{y'}(y'_n, y_n, x), \quad q_0(x) = -F_y(y'_n, y_n, x), \quad f_n(x) = F(y'_n, y_n, x) + q_0(x)\,y_n + q_1(x)\,y'_n.$$

Now, we implement the generalized pseudospectral method on equation (8) for each iteration of the pseudo-linearization method. We consider the approximate solution of the equation corresponding to equation (1) as follows:

$$y_{n+1}(x) \approx y_{N,n+1}(x) = \sum_{j=0}^{N} y_{N,n+1}(x_j) \; L_j^{\phi}(x) = \left(L^{\phi}(x)\right)^T Y_{N,n+1}. \tag{9}$$

where the vectors are defined as $Y_{N,n+1} = [y_{N,n+1}(x_0), \cdots, y_{N,n+1}(x_N)]^T$ and $L^{\phi}(x) = \left[L_j^{\phi}(x_0), \cdots, L_j^{\phi}(x_N)\right]^T$.

According to the collocation method, the residual error function for equation (8) in each iteration of the quasi-linearization method is defined as follows:

$$Res(x) = y_{N,n+1}''(x) + q_1(x) \; y_{N,n+1}'(x) + q_0(x) \; y_{N,n+1}(x) - f_n(x), \tag{10}$$

and we set it equal to zero in the interpolation points $\{x_i\}_{i=0}^{N}$ and also use the derivative operational matrices in Theorems 1 and 2 and equation (9). So, we have:

$$Res(x_i) = y_{N,n+1}''(x_i) + q_1(x_i) \; y_{N,n+1}'(x_i) + q_0(x_i) \; y_{N,n+1}(x_i) - f_n(x_i)$$

$$= \sum_{j=0}^{N} y_{N,n+1}(x_j) \, \mathbf{D}_{kj}^{(2)} + q_1(x_i) \sum_{j=0}^{N} y_{N,n+1}(x_j) \, \mathbf{D}_{kj}^{(1)} + q_0(x_i) y_{N,n+1}(x_i) - f_n(x_i) = 0,$$

and in matrix form

$$\mathbf{D}^{(2)} Y_{N,n+1} + Q_1 \, \mathbf{D}^{(1)} Y_{N,n+1} + Q_0 \, Y_{N,n+1} = F_n, \tag{11}$$

where the vector of $F_n = [f_n(x_0), \cdots, f_n(x_N)]^T$ and matrices $Q_j = Diag(q_j(x_0), \cdots, q_j(x_N))$ are defined for $j = 0, 1$.

To confirm the initial and boundary conditions of the equation, the several lines of equations are replaced in relation (11) with the initial and boundary conditions. Now, by solving the following linear system using a suitable method, the unknown vectors $Y_{N,n+1}$ is calculated, and the approximate solution in relation (9) is obtained for each iteration in the quasi-linearization method:

$$\begin{cases} \left(\mathbf{D}^{(2)} + Q_1 \, \mathbf{D}^{(1)} + Q_0\right) \, Y_{N,n+1} = F_n, \\ \left(L^{\phi}(0)\right)^T Y_{N,n+1} = A, \\ \left(\left.(L^{\phi}(x))'\right|_{x=0}\right)^T Y_{N,n+1} = B. \end{cases}$$

3. Numerical investigations

In this section, the presented method is examined on two nonlinear differential equations and compared the results with other methods to show the effectiveness of the method.

3.1. Catalytic reaction equation in a flat particle

In many engineering and industrial applications, catalytic processes in chemical reactors are often very useful. This has caused special attention to the study of catalytic reactions at the single particle level [14]. In addition, it has been almost two hundred years that the problem of how molecular particles diffusion and the behavior of porous catalytic particles has been studied [15, 16, 17]. Most chemical reactions are accompanied by heat transfer effects so most of them either release or absorb heat. This can lead to a significant increase (or decrease) in temperature towards the center of the particles [18, 19, 20]. Since chemical reactions grow very fast (exponentially) with temperature, this effect can change the behavior of catalyst particles in unexpected ways. The analysis of chemical kinetics with diffusion effects usually leads to the solution of high-order nonlinear differential equations, for further study reference [21] can be found. This issue has also been investigated by Mutsa and his colleagues using a numerical method [22].

Hlavacek et al. obtained a nonlinear differential equation with boundary conditions for the catalytic reaction in a flat particle as follows [23]:

$$\frac{d^2 y}{dx^2} - \lambda y e^{\left(\frac{\gamma \beta (1-y)}{1+\beta(1-y)}\right)} = 0, \tag{12}$$

with boundary conditions:

$$y'(0) = 0, \qquad y(1) = 1, \tag{13}$$

where y is the concentration of the reactant, x is the measurement coordinate in the direction of the particle, λ is Thiele's modulus or the reaction rate parameter, δ is the activation energy parameter that shows the sensitivity of the reaction rate to temperature, and β is the heat evolution parameter that is the maximum change It shows the temperature that can exist inside the particle compared to the boundary temperature.

Now, we solve the equation (12) for different values of its parameters using the proposed method. Considering that this equation is in the interval $[0, 1]$, a suitable choice for the values of a,b, the function of $\phi(x)$, interpolation and collocation points in Definition 1 is as $a = 0$, $b = 1$, $\phi(x) = \frac{2}{b-a}\left(x - \frac{b+a}{2}\right)$ and the roots of shift Chebyshev polynomials from order $N+1$ in the form $x_j = \frac{b+a}{2} - \frac{b-a}{2}\cos\left(\frac{2\pi j}{2(N+1)}\right)$ where $j = 0, \ldots, N$. Also, according to the boundary conditions of the problem, the initial guess function is chosen as $y_0(x) = \frac{1}{2}\left(x^2 + 1\right)$, which is satisfy in both boundary conditions.

Figure 1(a) shows the obtained plots from the residual errors corresponding to equation (10) using the proposed method on the values of $\lambda = 0/05$, $\beta = 0/4$, and $\gamma = 20$ with 10 repetitions of the pseudo-linearization method for different choices of N. As it is known, as the number of points increases, the amount of residual error decreases and this indicates the convergence of the proposed method.

Figure 1(b) shows the obtained plots from the solution of equation (12) using the proposed method for the values of $\lambda = 0/05$, $\beta = 0/4$, and $N = 25$ with 10 repetitions of the pseudo-linearization method for different choices of γ. As can be seen, the plots of the solutions decrease with the increase of the γ value.

Figure 1(c) shows the obtained plots from the solution of equation (12) using the proposed method on the values of $\gamma = 12$, $\beta = 0/4$, and $N = 25$ with 10 repetitions of the pseudo-linearization method for different choices of λ.

Figure 1(d) shows the obtained plots from the solution of equation (12) using the proposed method on the values of $\gamma = 12$, $\lambda = 0/3$, and $N = 25$ with 10 repetitions of the pseudo-linearization method for different choices of β.
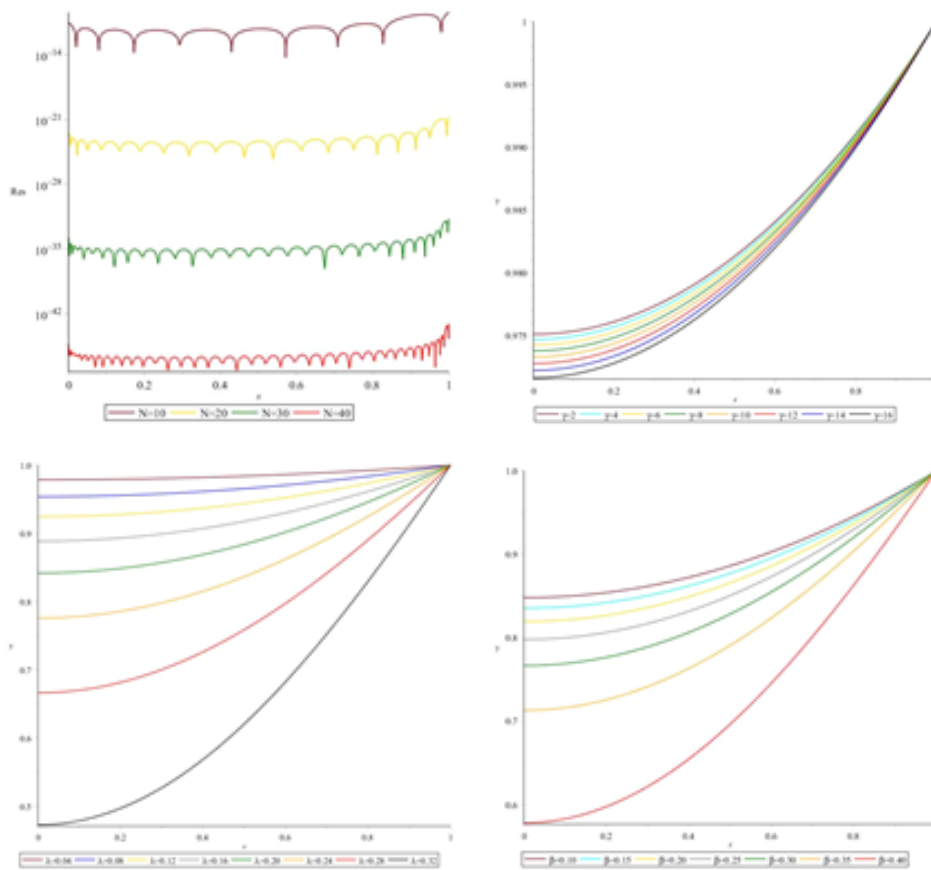

Figure 1. (a) Residual errors with different numbers of points to show the convergence of the method. (b) The solution plots with the different values of γ. (c) Plots of the solution of the equation with the different values of λ (d) Plots of the solution of the equation with the different values of β.

Table 2 shows the comparison of the obtained values for y(0) with the proposed method and the methods provided in Ref. [22] on the values of $\gamma = 12$, $\beta = 0/4$, and $N = 25$ with 10 repetitions of the pseudo-linearization method for the different choices of λ.

Table 3 shows the comparison of the obtained values for y(0) with the proposed method and the methods provided in Ref. [22] on the values of $\gamma = 12$, $\lambda = 0/3$, and $N = 25$ with 10 repetitions of the pseudo-linearization method for different choices of β.

Table 2. Comparison of the obtained values for y(0) with different choices of λ, in example 1

λ	Proposed method	Ref. [22]	Ref.[22]	Residual error
0.04	0.97861566251444542149	0.97861566	0.97861566	8.24e-34
0.08	0.95387919037288495067	0.95387919	0.95387919	2.37e-29
0.12	0.92454709646398678364	0.92454710	0.92454710	1.94e-26
0.16	0.88852609004680344510	0.88852609	0.88852609	4.11e-25
0.20	0.84188248981647992909	0.84188249	0.84188249	8.68e-23
0.24	0.77590839119657396323	0.77590839	0.77590839	1.54e-20
0.28	0.66638659712550789137	0.66638660	0.66638660	2.84e-18
0.32	0.47282849827021284758	0.47282850	0.47282850	1.53e-16

Table 3. Comparison of the obtained values for y(0) with different choices of β, in example 1

β	Proposed method	Ref. [22]	Ref. [22]	Residual error
0.10	0.84788700681997285470	0.84788701	0.84788701	5.58e-28
0.15	0.83529530923946625653	0.83529531	0.83529531	2.92e-26
0.20	0.81926631249860521567	0.81926631	0.81926631	1.68e-24
0.25	0.79776695788568956012	0.79776696	0.79776696	5.07e-23
0.30	0.76641380434196407094	0.76641380	0.76641380	2.70e-21
0.35	0.71291150352130587507	0.71291150	0.71291150	2.27e-20
0.40	0.57812876564004366356	0.57812877	0.57812877	4.92e-17

Table 4 shows the comparison of the obtained values for y(0) with the proposed method and the methods provided in Ref. [22] on the values of $\beta = 0/4$, $\lambda = 0/05$, and $N = 25$ with 10 repetitions of the pseudo-linearization method for different choices of γ.

Table 4. Comparison of the obtained values for y(0) with different choices of γ, in example 1

γ	Proposed method	Ref. [22]	Ref. [22]	Residual error
2	0.97511280488111687487	0.7511280	0.97511280	1.91e-39
4	0.97469523927581541732	0.97469524	0.97469524	2.26e-38
6	0.97425591453548614356	0.97425591	0.97425591	2.95e-36
8	0.97379272786392154102	0.97379273	0.97379273	7.34e-35
1	0.97330326446373009355	0.97330326	0.97330326	5.26e-34
12	0.97278473289493909123	0.97278473	0.97278473	2.18e-32
14	0.97223388274346480432	0.97223388	0.97223388	1.49e-31
16	0.97164689842757845833	0.97164690	0.97164690	3.81e-30

3.2. Troesch's equation

Troesch's equation is a nonlinear differential equation with two boundary conditions, which is defined as follows [24]:

$$y''(x) = M \sinh(M\,y(x)), \qquad\qquad x \in [0,1], \qquad\qquad (14)$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 1, \qquad\qquad (15)$$

where M is a positive constant.

This equation occurs in the investigation of the limitation of a plasma column by radiation pressure [25] and also in the theory of porous gas electrodes [26, 27]. Some researchers have investigated this problem using approximate and numerical methods, for example, Nabati and Jalalund by the exponential Sinc-Galerkin method [24], Chang by the shooting method [28], Feng et al. by the modified Homotopy methods [29], Zarabnia et al. by the Sinc-Galerkin method [30], EL-Gamel by the Sinc-Collocation method [31], Deeba et al. with the Adomin decomposition method [32], Saadatmandi and Abdolahi by collocation method based on Christov basis rational functions [33], Khuri and Sayfy by the B-spline method [34], and Delkhosh and Parand in [1].

Now, we solve the equation (14) for different values of M using the proposed method. Considering that this equation is in the interval [0, 1], a suitable choice for the values of a, b, interpolation and collocation points in Definition 1 is as $a = 0$, $b = 1$, and the roots of shift Chebyshev polynomials from order $N + 1$. Also, according to the boundary conditions of the problem, the initial guess function is chosen as $y_0(x) = x$, which is satisfy in both boundary conditions.

Considering the form of the problem, which includes an exponential function, the structure of its solution may be exponential. For this reason, we examine this problem for two different choices of the function $\phi(x)$, polynomial $\phi(x) = \frac{2}{b-a}\left(x - \frac{b+a}{2}\right)$ and exponential $\phi(x) = e^x$. Figure 2 shows the plots of the residual error function for these two choices of the $\phi(x)$ function on $M = 0/5, 1, 2, 3$ values. As can be seen, for values of M that are less than 1, the polynomial function (red plot) is a more suitable choice than the exponential function, but for values greater than 1, the exponential function (blue plot) is a more suitable choice. For this reason, in the following, the problem is solved depending on the value of M for two different choices of the $\phi(x)$ function (polynomial and exponential).

Figure 3 shows the plots related to the solution of equation (14) obtained from the proposed method for the different choices of the value of M.

Table 5 shows the comparison of the obtained values for the solution of equation (14) with the proposed method and the methods presented in other references for $M = 0/5$ and $N = 40$ with 20 repetitions of the pseudo-linearization method.
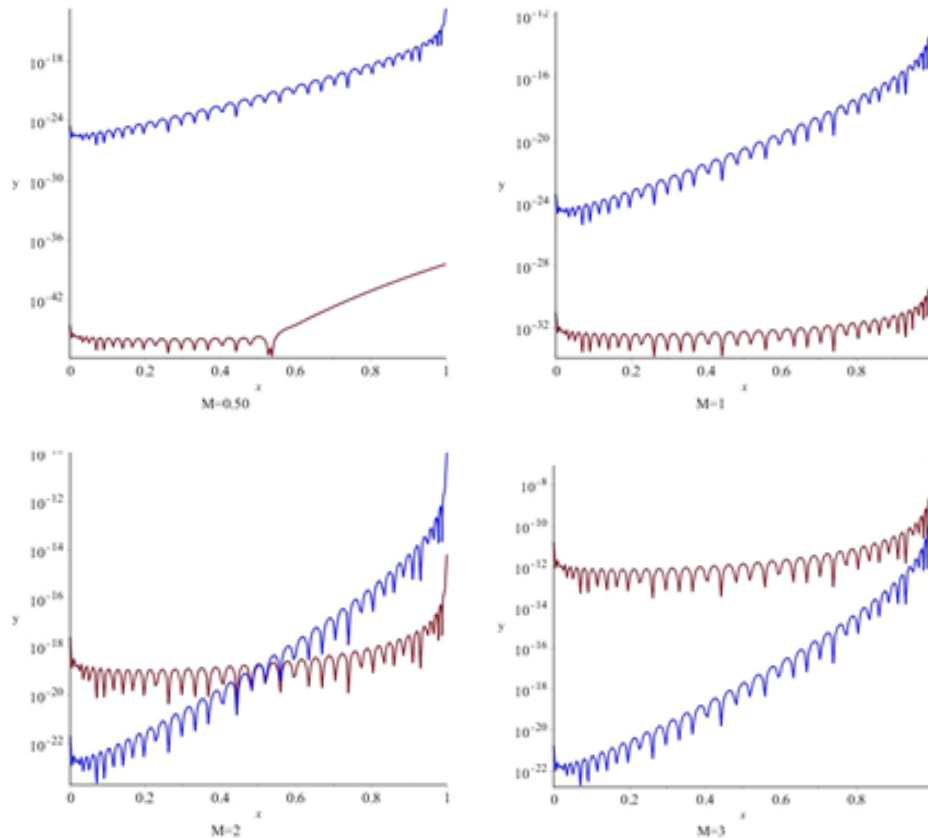

Figure 2. Plots of the residual error function for two different choices of the function $\phi(x)$ on several values of M, in example 2

Table 5. Comparison of the obtained values for the solution equation with $M = 0.5$, in example 2

| x	Proposed method	Ref. [24]	Ref. [32]	Ref. [29]	Residual error
0.1	0.095944349292286724119147517360	0.09594434932	0.09593835	0.09593956	1.43e-46
0.2	0.192128747660289190772447934540	0.19212874768	0.19211805	0.19211932	4.11e-47
0.3	0.288794400893448546519657729100	0.28879440094	0.28878032	0.28878069	5.29e-47
0.4	0.386184846362337310994924094600	0.38618484638	0.38616870	0.38616754	5.82e-47
0.5	0.484547164744892516751951647470	0.48454716477	0.48453029	0.48452741	2.17e-46
0.6	0.584133248445574184548390016750	0.58413324848	0.58411697	0.58411278	3.13e-45
0.7	0.685201148301847334813051658990	0.68520114831	0.68518684	0.68518224	2.09e-43
0.8	0.788016522649566667916575735910	0.78801652269	0.78800556	0.78800183	8.53e-42
0.9	0.892854216136313718302953269108	0.89285421616	0.89284802	0.89284621	2.26e-40
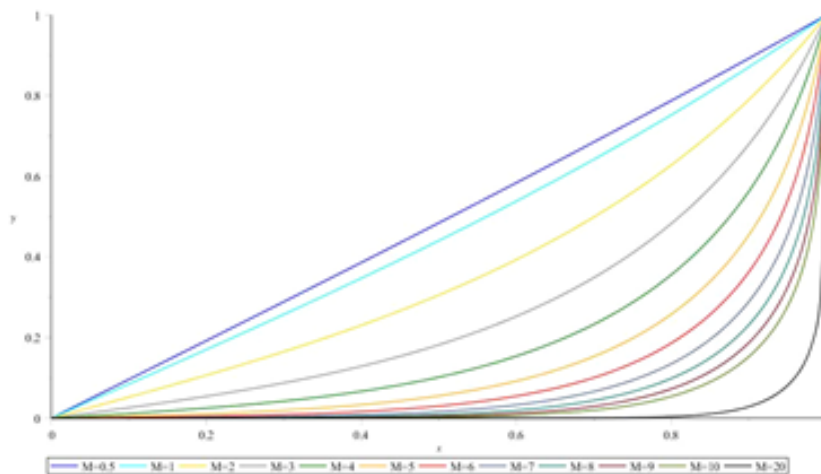
Figure 3. Plots of the solution of the equation with different choices of M, in example 2

Table 6 shows the comparison of the obtained values for the solution of equation (14) with the proposed method and the methods presented in other references for $M = 1$ and $N = 40$ with 20 repetitions of the pseudo-linearization method.

Table 6. Comparison of the solution values of the equation in the case of $M = 1$, in example 2

x	Proposed method	Ref. [24]	Ref. [32]	Ref. [29]	Residual error
0.1	0.084661256551567723530367824	0.084661256642	0.08424876	0.0843817004	4.26e-33
0.2	0.170171358177549610678777838	0.170171358273	0.16943070	0.1696207644	1.25e-33
0.3	0.257393908079888201621258300	0.257393908175	0.25641450	0.2565929224	1.65e-33
0.4	0.347222855110497580836420532	0.347222855224	0.34608572	0.3462107378	1.87e-33
0.5	0.440599835168425203342680680	0.440599835276	0.43940198	0.4394422743	5.96e-33
0.6	0.538534398076897487589758370	0.538534398177	0.53736570	0.5373300622	3.15e-33
0.7	0.642128609190826787111592440	0.642128609348	0.64108380	0.6410104651	4.84e-33
0.8	0.752608094046386963178172110	0.752608094135	0.75178800	0.7517335467	7.09e-32
0.9	0.871362519798188737239482490	0.871362519949	0.87090870	0.8708835371	6.17e-28

Table 7 shows the comparison of the obtained values by the proposed method and the methods presented in other references for $M = 5$.

Table 7. Comparison of the solution values of the equation in the case of $M = 5$, in example 2

x	Proposed method	Ref. [24]	Ref. [31]	Ref. [34]	Residual error
0.2	0.010753406642556	0.01075340	0.00762552	0.01002027	9.51e-23
0.4	0.033200490979016	0.03320049	0.03817903	0.03099793	3.33e-18
0.8	0.258216487315564	0.25821648	0.23252435	0.24170496	1.75e-13
0.9	0.455060027382638	0.45506002	0.44624551	0.42461830	1.34e-09

4. Conclusion

In this study, a new hybrid numerical method using generalized pseudospectral and Newton-Kantorovich quasi-linearization methods is presented to solve nonlinear differential equations. Since derivative operational matrices are provided for generalized Lagrange functions, and also any nonlinear equation can be converted into a sequence of linear equations, in the implementation of this method, there is no need to calculate the derivative mathematically and the calculation costs are reduced. The proposed method has been implemented to solve two practical problems and the results show that this method is efficient, useful and accurate and can be easily used on other differential equations.

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