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Some types of filters in equality algebras

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. Equality algebras were introduced by S. Jenei as a possible algebraic semantic for fuzzy type theory. In this paper, we introduce some types of filters such as (positive) implicative, fantastic, Boolean, and prime filters in equality algebras and we prove some results which determine the relation between these filters. We prove that the quotient equality algebra induced by an implicative filter is a Boolean algebra, by a fantastic filter is a commutative equality algebra, and by a prime filter is a chain, under suitable conditions. Finally, we show that positive implicative, implicative, and Boolean filters are equivalent on bounded commutative equality algebras.

1 Introduction

Every many valued logic is uniquely determined by the algebraic properties of the structure of its truth values. The well-known algebraic structure is residuated lattice. BL-algebras, MTL-algebras, MV-algebras, and so forth, are the best known classes of residuated lattices [4, 5]. Since the algebra of truth values is no longer a residuated lattice, a specific algebra is introduced and called an EQ-algebra [11] by Novák and De Baets. EQ-algebras

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generalize the residuated lattices which have three binary operations, meet, multiplication, fuzzy equality, and a unit element. As it was mentioned in [7], if the product operation in EQ-algebras is replaced by another binary operation smaller or equal than the original product we will still obtain an EQ-algebra, and this fact might make it difficult to obtain certain algebraic results. For this reason, a new structure, called *equality algebra* similar to EQ-algebras, but without a product, was introduced by S. Jenei in [7]. These algebras are assumed for a possible algebraic semantics of fuzzy type theory. From these points of view, it is meaningful to study equality algebras. Filter theory plays an important role in studying these algebras. From a logical point of view, various filters correspond to various sets of provable formulas. The sets of provable formulas can be described by fuzzy filters of those algebraic semantics. Up to now, some types of filters on special residuated lattices based on logical algebras have been widely studied and some important results have been obtained, [1, 6, 12, 13]. In [10, 11], the notions of a prefilter (which coincides with filters in residuated lattices), a prime prefilter, and a (positive) implicative prefilter in EQ-algebras were proposed and some characterizations of them have been investigated. It is proved that implicative prefilters and positive implicative prefilters are equivalent on “good” IEQ-algebras and the quotient algebras induced by positive implicative filters in residuated EQ-algebras are idempotent residuated EQ-algebras. Hence, the properties of filters have a strong influence on the structure properties of algebras. Every equality algebra is EQ-algebra. S. Jenei and L. Kóródi proposed the notions of filters and congruence relations in equality algebras and obtained some of their properties [8]. L.C. Ciungu proved that in any equality algebra, there is a one-to-one correspondence between the set of all congruence relations and the set of all filters [3]. In this paper, we introduce the notions of (positive) implicative, fantastic, Boolean, and prime filters in equality algebras inspired by [6, 10, 13]. We investigate some characterizations of these filters and we prove that the quotient algebra induced by a fantastic filter in an equality algebra is a commutative equality algebra, by an implicative filter in bounded equality algebra is a Boolean algebra, and by a prime filter in prelinear equality algebras is a chain. Moreover, the relation between these filters is discussed. It is proved that positive implicative filters, implicative filters, and Boolean filters are equivalent on bounded commutative equality algebras.

2 Preliminaries

In this section, we recollect some definitions and results which will be used in this paper and we shall not cite them every time they are used.

Definition 2.1. [7] An algebra $\mathcal{E} = (E, \wedge, \sim, 1)$ of the type $(2, 2, 0)$ is called an *equality algebra* if it satisfies the following conditions, for all $x, y, z \in E$:
(E1) $(E, \wedge, 1)$ is a meet-semilattice with the top element 1,
(E2) $x \sim y = y \sim x$,
(E3) $x \sim x = 1$,
(E4) $x \sim 1 = x$,
(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called *meet* (*infimum*) and $\sim$ is an *equality operation*. We write $x \leq y$ if and only if $x \wedge y = x$, for all $x, y \in E$. Also, two other operations are defined and called *implication* and *equivalence operation*, respectively:

$$x \to y = x \sim (x \wedge y) \tag{I}$$

$$x \leftrightarrow y = (x \to y) \wedge (y \to x) \tag{II}$$

If $\sim$ coincides with $\leftrightarrow$, then an equality algebra is called *equivalential*. An equality algebra $(E, \sim, \wedge, 1)$ is *bounded* if there exists an element $0 \in E$ such that $0 \leq x$, for all $x \in E$. In a bounded equality algebra E, we define the negation " $'$ " on E by, $x' = x \to 0 = x \sim 0$, for all $x \in E$. If $(x')' = x$, for all $x \in E$, then the bounded equality algebra E is called *involutive*. An equality algebra E is called *prelinear* if 1 is the unique upper bound of the set $\{x \to y, y \to x\}$, for all $x, y \in E$. An equality algebra E is called *commutative* if $(x \to y) \to y = (y \to x) \to x$, for all $x, y \in E$. A *lattice equality algebra* is an equality algebra which is a lattice, as well.

Definition 2.2. [8] Let $\mathcal{E} = (E, \sim, \wedge, 1)$ be an equality algebra and F be a non-empty subset of E. Then F is called a *deductive system* or a *filter* of $\mathcal{E}$ if, for all $x, y \in E$, we have
(i) $1 \in F$,
(ii) if $x \in F$ and $x \leq y$, then $y \in F$,

(iii) if $x \in F$ and $x \sim y \in F$, then $y \in F$.
Denote by $F(\mathcal{E})$ the set of all filters of $\mathcal{E}$. Clearly, $F(\mathcal{E})$ is closed under arbitrary intersections and $\{1\} \in F(\mathcal{E})$, so $(\mathcal{E}, \subseteq)$ is a complete lattice. A filter F of $\mathcal{E}$ is called a *proper filter* if $F \neq E$. It can be easily seen that, if $\mathcal{E}$ is a bounded equality algebra, then a filter is proper if and only if it does not contain 0. A proper filter of $\mathcal{E}$ is called a *maximal filter* if it is not properly contained in any other proper filter of $\mathcal{E}$.

Proposition 2.3. [3, 8] *Let $\mathcal{E}$ be an equality algebra. Then, $F \in F(\mathcal{E})$ if and only if, for all $x, y \in E$,*
(i) $1 \in F$,
(ii) *if x and $x \to y \in F$, then $y \in F$.*

Definition 2.4. [8] Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra. A subset Θ of $E \times E$ is called a *congruence relation* on E, if it is an equivalence relation on E and, for all $x, y, z, w \in E$ such that (x, z), $(y, w) \in \Theta$, we have $(x \wedge y, z \wedge w) \in \Theta$ and $(x \sim y, z \sim w) \in \Theta$. Denote by $Con(\mathcal{E})$ the set of all congruence relations on E.

Proposition 2.5. [3, 8] *If $\mathcal{E}$ is an equality algebra, $F \in F(\mathcal{E})$ and the relations $\Theta_{\overrightarrow{F}}$ and Θ_F on E are defined by*

$$(x, y) \in \Theta_{\overrightarrow{F}} \Leftrightarrow \{x \to y, y \to x\} \subseteq F \quad \text{and} \quad (x, y) \in \Theta_F \Leftrightarrow x \sim y \in F,$$

then Θ_F, $\Theta_{\overrightarrow{F}} \in Con(\mathcal{E})$, and $\Theta_{\overrightarrow{F}} = \Theta_F$.

Let $E/F = \{[x] \mid x \in E\}$, where $[x] = \{y \in E \mid (x, y) \in \Theta_F\}$. Then the binary relation $\leq_F$ on E/F which is defined by

$$[x] \leq_F [y] \text{ if and only if } x \to y \in F$$

is an order relation on E/F.

Theorem 2.6. [3] *Let $\mathcal{E}$ be an equality algebra. Then there is a one-to-one correspondence between $F(\mathcal{E})$ and $Con(\mathcal{E})$.*

Theorem 2.7. [3] *Let $(E, \wedge, \sim, 1)$ be an equality algebra and $F \in F(\mathcal{E})$. Then $(E/F, \sim_F, \wedge_F, 1_F)$ is an equality algebra, where for every $x, y \in E$,*

$$1_F = [1], \; [x] \sim_F [y] = [x \sim y], \; [x] \wedge_F [y] = [x \wedge y].$$

The following propositions provide some properties of equality algebras.

Proposition 2.8. [7] *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra. Then the following properties hold, for all $x, y, z \in E$:*
(i) $x \to y = 1$ *if and only if* $x \leq y$,
(ii) $1 \to x = x$, $x \to 1 = 1$, $x \to x = 1$,
(iii) $x \leq y \to x$,
(iv) $x \leq (x \to y) \to y$,
(v) $x \to y \leq (y \to z) \to (x \to z)$,
(vi) $x \leq y \to z$ *if and only if* $y \leq x \to z$,
(vii) $x \to (y \to z) = y \to (x \to z)$.

Proposition 2.9. [14] *Let $\mathcal{E} = (E, \wedge, \sim, 1)$ be an equality algebra. Then, for all $x, y, z \in E$, the following statements hold:*
(i) $x \leq y$ *implies* $y \to z \leq x \to z$, $z \to x \leq z \to y$,
(ii) $x \to y = x \to (x \wedge y)$,
(iii) $x \to y \leq (z \to x) \to (z \to y)$,
(iv) $x \to y \leq (x \wedge z) \to (y \wedge z)$.

Proposition 2.10. [14] *Let E be a lattice equality algebra. Then, for all $x, y \in E$, the following statements hold:*
(i) $x \to y = (x \vee y) \to y$,
(ii) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

Proposition 2.11. [14] *Let E be a bounded lattice equality algebra. Then, for all $x, y \in E$, the following statements hold:*
(i) $x \leq (x')'$,
(ii) $(x \vee y)' = x' \wedge y'$.

Theorem 2.12. [14] *Every commutative equality algebra is a lattice.*

Theorem 2.13. [14] *Any prelinear equality algebra is a distributive lattice.*

Theorem 2.14. [14] *Let $\mathcal{E} = (E, \wedge, \sim, 0, 1)$ be a bounded commutative equality algebra. Then $\Phi(\mathcal{E}) = (E, \oplus, *, 0)$ is an MV-algebra, where the operations $\oplus$ and $*$ defined as $x \oplus y = x' \to y$, $x^* = x'$, and $\to$ denotes the implication of $\mathcal{E}$.*

Notation. From now on, E denotes an *equality algebra*, unless otherwise stated.

3 (Positive) Implicative filters in equality algebra

In this section, we give some characterizations of (positive) implicative filters and investigate the relations between them.

Definition 3.1. Let F be a non-empty subset of E such that $1 \in F$. Then F is called a *positive implicative filter* if $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$, for all $x, y, z \in E$.

The following examples show that positive implicative filters in equality algebras exist.

Example 3.2. Let $(E = \{0, a, b, 1\}, \leq)$ be a chain. Define the operations $\sim$ and $\to$ on E by

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	b
1	0	a	b	1

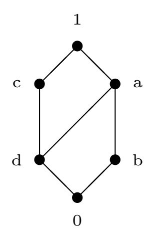
$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

By routine calculations, we can see that $(E, \wedge, \sim, 1)$ is an equality algebra and $F = \{1, b\}$ is a positive implicative filter of E.

Lemma 3.3. *Any positive implicative filter of E is a filter.*

Proof. Let x, $x \to y \in F$. Then $1 \to (x \to y) = x \to y \in F$ and $1 \to x = x \in F$. Since F is a positive implicative filter, $y = 1 \to y \in F$. Hence, F is a filter of E. $\square$

Example 3.4. Let $(E = \{0, a, b, c, d, 1\}, \leq)$ be a lattice with the following diagram. Define the operations $\sim$ and $\to$ on E by



$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then, by routine calculations, we can see that $(E, \wedge, \sim, 0, 1)$ is an equality algebra and $F = \{1, c\}$ is a filter, but it is not a positive implicative filter. Because $a \to (a \to 0) = c \in F$ and $a \to a = 1 \in F$, but $a \to 0 = d \notin F$.

Proposition 3.5. *Let F be a non-empty subset of E. Then, for all $x, y, z \in E$, the following statements are equivalent*:

(i) *F is a positive implicative filter of E,*

(ii) *F is a filter and if $x \to (x \to y) \in F$, then $x \to y \in F$,*

(iii) *F is a filter and if $z \to (y \to x) \in F$, then $(z \to y) \to (z \to x) \in F$,*

(iv) *$1 \in F$ and if $z \in F$ and $z \to (x \to (x \to y)) \in F$, then $x \to y \in F$.*

Proof. (i) $\Rightarrow$ (ii) Let F be a positive implicative filter. Then, by Lemma 3.3, F is a filter of E. If $x \to (x \to y) \in F$, since $x \to x = 1 \in F$ and F is a positive implicative filter, $x \to y \in F$.

(ii) $\Rightarrow$ (iii) Let $z \to (y \to x) \in F$. By Propositions 2.8(vii), 2.9(i) and (iii),

$$z \to (z \to ((z \to y) \to x)) = z \to ((z \to y) \to (z \to x)) \geq z \to (y \to x).$$

Since F is a filter and $z \to (y \to x) \in F$, we get $z \to (z \to ((z \to y) \to x)) \in F$. Then, by assumption $z \to ((z \to y) \to x) \in F$. Thus, by Proposition 2.8(vii), $(z \to y) \to (z \to x) \in F$.

(iii) $\Rightarrow$ (iv) Since F is a filter, $1 \in F$. If $z \in F$ and $z \to (x \to (x \to y)) \in F$, then $x \to (x \to y) \in F$. By assumption, $(x \to x) \to (x \to y) \in F$, and so $x \to y \in F$.

(iv) $\Rightarrow$ (i) Let $z \to (y \to x) \in F$ and $z \to y \in F$. By Proposition 2.9(iii),

$$z \to (y \to x) \leq (z \to y) \to (z \to (z \to x)).$$

Since F is a filter and $z \to (y \to x) \in F$, we have $(z \to y) \to (z \to (z \to x)) \in F$. Then, by (iv), $z \to x \in F$. $\square$

Lemma 3.6. *Let F be a filter of E. Then the following properties hold:*
(i) $x \sim y \in F$ *and* $y \sim z \in F$ *imply* $x \sim z \in F$,
(ii) $x \to y \in F$ *and* $y \to z \in F$ *imply* $x \to z \in F$.

Proof. (i) If $x \sim y \in F$ and $y \sim z \in F$, then by (E7) and (E2), $x \sim y \leq (y \sim z) \sim (x \sim z)$. Since F is a filter, $(y \sim z) \sim (x \sim z) \in F$, and so $x \sim z \in F$.

(ii) The proof is similar to the proof of (i). □

Proposition 3.7. *Let F be a filter of E. Then F is a positive implicative filter if and only if, for all $x, y \in E$, $(x \wedge (x \to y)) \to y \in F$.*

Proof. ($\Rightarrow$) Since $x \wedge (x \to y) \leq x \to y$ and $x \wedge (x \to y) \leq x$, we have $(x \wedge (x \to y)) \to (x \to y) = 1 \in F$ and $(x \wedge (x \to y)) \to x = 1 \in F$. Since F is a positive implicative filter, $(x \wedge (x \to y)) \to y \in F$.

($\Leftarrow$) Let $x \to (y \to z) \in F$ and $x \to y \in F$. By Proposition 2.9(iv), $x \to (y \to z) \leq (x \wedge y) \to (y \wedge (y \to z))$. Since F is a filter, $(x \wedge y) \to (y \wedge (y \to z)) \in F$. By Proposition 2.9(ii), $x \to (x \wedge y) = x \to y \in F$, and so by Lemma 3.6(ii), $x \to (y \wedge (y \to z)) \in F$. By assumption, $(y \wedge (y \to z)) \to z \in F$. Thus, by Lemma 3.6(ii), $x \to z \in F$. □

Corollary 3.8. *Suppose F and G are two filters of E and $F \subseteq G$. If F is a positive implicative filter, then G is a positive implicative filter, too.*

Proof. By Proposition 3.7, the proof is clear. □

Corollary 3.9. *Every filter of E is a positive implicative filter if and only if $\{1\}$ is a positive implicative filter.*

Proposition 3.10. *Let F be a filter of E. Then F is a positive implicative filter if and only if every filter of equality algebra E/F is a positive implicative filter.*

Proof. Let F be a filter of E. Then by Proposition 3.7, F is a positive implicative filter of E if and only if for any $x, y \in E$, $(x \wedge (x \to y)) \to y \in F$ if and only if $[(x \wedge (x \to y)) \to y] = [1]$ if and only if, for every $x, y \in E$, $([x] \wedge_F ([x] \to_F [y])) \to_F [y] = [1]$ if and only if, by Proposition 3.7, $\{[1]\}$ is a positive implicative filter of E/F if and only if, by Corollary 3.9, every filter of E/F is a positive implicative filter. □

Proposition 3.11. *Let F be a non-empty subset of E. Then F is a positive implicative filter if and only if, for any $a \in E$, $F_a = \{x \in E \mid a \to x \in F\}$ is the least filter of E containing F and $\{a\}$.*

Proof. ($\Rightarrow$) Suppose F is a positive implicative filter of E and $a \in E$. Since $a \to 1 = 1 \in F$, $1 \in F_a$. If $x, x \to y \in F_a$, then $a \to x \in F$ and $a \to (x \to y) \in F$. Since F is a positive implicative filter, we have $a \to y \in F$, and so $y \in F_a$. Therefore, F_a is a filter of E. Now, let $x \in F$. By Proposition 2.8(iii), $x \leq a \to x$ and $x \in F$, then $a \to x \in F$, and so, $x \in F_a$. Hence, $F \subseteq F_a$. Moreover, $a \to a = 1 \in F$, then $a \in F_a$. Thus, $F \cup \{a\} \subseteq F_a$. Let G be a filter of E such that $F \cup \{a\} \subseteq G \subseteq F_a$. Then $a \to x \in F \subseteq G$, for every $x \in F_a$. Since G is a filter and $a \in G$, $x \in G$. Thus $F_a \subseteq G$, and so $F_a = G$. Hence, F_a is the least filter of E containing F and $\{a\}$.

($\Leftarrow$) Let $x, y \in E$. If $x \to (x \to y) \in F$, then $x \to y \in F_x$. Since F_x is a filter containing $\{x\}$, we have $y \in F_x$, and so $x \to y \in F$. Thus, by Proposition 3.5(ii), F is a positive implicative filter. □

Definition 3.12. Let F be a non-empty subset of E. Then F is called an *implicative filter* if $1 \in F$ and if $z \to ((x \to y) \to x) \in F$ and $z \in F$, then $x \in F$, for all $x, y, z \in E$.

The following example shows that implicative filter in equality algebras exists.

Example 3.13. Let $(E = \{0, a, b, c, 1\}, \leq)$ be a chain. Define the operations $\sim$ and $\to$ on E by

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	b	b	a
b	0	b	1	c	b
c	0	b	c	1	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	c	1	1
1	0	a	b	c	1

Then by routine calculations, we can see that $(E, \sim, \wedge, 1)$ is an equality algebra and $F = \{1, a, b, c\}$ is an implicative filter of E.

Lemma 3.14. *Any implicative filter of E is a filter.*

Proof. Let $x, x \to y \in F$. Then $x \to y = x \to ((y \to 1) \to y) \in F$. Since F is an implicative filter, $y \in F$. Hence, F is a filter. $\square$

Example 3.15. In Example 3.2, $F = \{1, b\}$ is a filter which is not an implicative filter. Because $1 \to ((a \to 0) \to a) \in F$, but $a \notin F$.

Proposition 3.16. *Let F be a filter of bounded equality algebra E. For all $x, y, z \in E$, the following statements are equivalent:*
(i) *F is an implicative filter of E,*
(ii) *$(x \to y) \to x \in F$ implies $x \in F$,*
(iii) *$x' \to x \in F$ implies $x \in F$,*
(iv) *$x \to (z' \to y) \in F$ and $y \to z \in F$ imply $x \to z \in F$,*
(v) *$x \to (y' \to y) \in F$ implies $x \to y \in F$.*

Proof. (i) $\Rightarrow$ (ii) Let $(x \to y) \to x \in F$. Then, by Proposition 2.8(ii), $(x \to y) \to x = 1 \to ((x \to y) \to x) \in F$ and $1 \in F$. Since F is an implicative filter, $x \in F$.
(ii) $\Rightarrow$ (i) Suppose $z \in F$ and $z \to ((x \to y) \to x) \in F$. Since F is a filter of E, $(x \to y) \to x \in F$ and by (ii), $x \in F$.

(ii) $\Rightarrow$ (iii) If $x' \to x \in F$, then $x' \to x = (x \to 0) \to x \in F$ and by (ii), $x \in F$.

(iii) $\Rightarrow$ (ii) Let $(x \to y) \to x \in F$. Since $0 \leq y$, by Proposition 2.9(i), $(x \to y) \to x \leq (x \to 0) \to x = x' \to x$. Since F is a filter of E, $x' \to x \in F$, and so by (iii), $x \in F$.

(iii) $\Rightarrow$ (iv) Let $x \to (z' \to y) \in F$ and $y \to z \in F$. By Proposition 2.8(vii) and (vi),

$$x \to (z' \to y) = z' \to (x \to y) \leq ((x \to y) \to (x \to z)) \to (z' \to (x \to z)).$$

Since F is a filter, $((x \to y) \to (x \to z)) \to (z' \to (x \to z)) \in F$. Since $y \to z \leq (x \to y) \to (x \to z)$ and $y \to z \in F$, $(x \to y) \to (x \to z) \in F$. Thus, by Proposition 2.3, $z' \to (x \to z) \in F$. Since $z' \to (x \to z) \leq (x \to z)' \to (x \to z)$, $(x \to z)' \to (x \to z) \in F$, and so by (iii), $x \to z \in F$.

(iv) $\Rightarrow$ (v) If $x \to (y' \to y) \in F$, then it is enough to choose $z = y$ in (iv). Since $y \to y = 1 \in F$, $x \to y \in F$.

(v) $\Rightarrow$ (iii) If $x' \to x \in F$, then $1 \to (x' \to x) \in F$. Thus, by (v), $x = 1 \to x \in F$. $\square$

Proposition 3.17. *Let F be a filter of bounded equality algebra E. Then F is an implicative filter if and only if $(x' \to x) \to x \in F$, for any $x \in E$.*

Proof. ($\Rightarrow$) Let $\alpha = (x' \to x) \to x$, for $x \in F$. Then

$$
\begin{aligned}
\alpha' \to \alpha =& (\alpha \to 0) \to \alpha \\
&= (((x' \to x) \to x) \to 0) \to ((x' \to x) \to x), \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by Proposition 2.8(vii)} \\
&= (x' \to x) \to ((((x' \to x) \to x) \to 0) \to x), \text{ by Proposition 2.8(v)} \\
&\geq (((x' \to x) \to x) \to 0) \to x' \\
&= (((x' \to x) \to x) \to 0) \to (x \to 0), \quad \text{by Proposition 2.8(v)} \\
&\geq x \to ((x' \to x) \to x), \quad \text{by Proposition 2.8(vii)} \\
&= (x' \to x) \to (x \to x) \\
&= (x' \to x) \to 1 \\
&= 1.
\end{aligned}
$$

Hence, $\alpha' \to \alpha \in F$, and so, by Proposition 3.16(iii), $\alpha \in F$.

($\Leftarrow$) Suppose $(x \to y) \to x \in F$. Since $0 \leq y$, by Proposition 2.9(i), $x \to 0 \leq x \to y$. Thus, $(x \to y) \to x \leq (x \to 0) \to x = x' \to x$. Since $(x \to y) \to x \in F$ and F is a filter of E, $x' \to x \in F$. Also, by assumption, $(x' \to x) \to x \in F$. Then, $x \in F$. Thus, by Proposition 3.16(ii), F is an implicative filter. □

In the following, we investigate the relation between positive and implicative filters.

Theorem 3.18. *Any implicative filter of E is a positive implicative filter.*

Proof. By Proposition 3.7, it is enough to prove that $(x \wedge (x \to y)) \to y \in F$, for all $x, y \in E$. For this, $x \wedge (x \to y) \leq x$, $x \wedge (x \to y) \leq x \to y$ and, by Proposition 2.9(i), $x \wedge (x \to y) \leq x \to y \leq (x \wedge (x \to y)) \to y$. Thus, $((x \wedge (x \to y)) \to y) \to y \leq (x \wedge (x \to y)) \to y$, and so $(((x \wedge (x \to y)) \to y) \to y) \to ((x \wedge (x \to y)) \to y) = 1 \in F$. Since F is an implicative filter, by Proposition 3.16(ii), $(x \wedge (x \to y)) \to y \in F$. □

The following example shows that not every positive implicative filter of E is an implicative.

Example 3.19. In Example 3.2, $F = \{1, b\}$ is a positive implicative filter which is not an implicative filter. Because, $(a \to 0) \to a = 1 \in F$, but $a \notin F$.

Theorem 3.20. *Let F be a positive implicative filter of bounded equality algebra E. Then F is an implicative filter if and only if $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$.*

Proof. ($\Rightarrow$) Let F be an implicative filter of E and $(x \to y) \to y \in F$, for any $x, y \in E$. Since $x \leq (y \to x) \to x$, by Proposition 2.9(i), $((y \to x) \to x)' \leq x' = x \to 0 \leq x \to y$. Then, by Proposition 2.9(i), $(x \to y) \to y \leq ((y \to x) \to x)' \to y$. Since $y \leq (y \to x) \to x$, $((y \to x) \to x)' \to y \leq ((y \to x) \to x)' \to ((y \to x) \to x)$. Thus, $(x \to y) \to y \leq ((y \to x) \to x)' \to ((y \to x) \to x)$. By Lemma 3.14, F is a filter, then $((y \to x) \to x)' \to ((y \to x) \to x) \in F$. Hence, by Proposition 3.16(iii), $(y \to x) \to x \in F$.

($\Leftarrow$) Let F be a positive implicative filter and $x' \to x \in F$. By Propositions 2.11(i) and 2.9(i), $x' \to x \leq x' \to x''$. Since F is a filter, $x' \to x'' \in F$. By Proposition 3.5(ii), $x' \to 0 \in F$. By assumption, we have $(0 \to x) \to x = x \in F$. Hence, by Proposition 3.16(iii), F is an implicative filter. $\square$

Corollary 3.21. *Let F be a positive implicative filter of bounded equality algebra E. Then F is an implicative filter if and only if $x'' \in F$ implies $x \in F$.*

Corollary 3.22. *In every involutive equality algebra, implicative filters and positive implicative filters coincide.*

Lemma 3.23. *If E is a bounded lattice equality algebra and for every $x \in E$, $x \vee x' = 1$, then $x \wedge x' = 0$.*

Proof. Let $x \vee x' = 1$. Then, by Proposition 2.11(i) and (ii), $x' \wedge x \leq x' \wedge x'' = (x \vee x')' = 1' = 0$. Thus, $x \wedge x' = 0$. $\square$

Notation. If E is involutive, then the converse of Lemma 3.23 holds. Let $x \wedge x' = 0$. Then, by Proposition 2.11(ii), $x \vee x' = (x \vee x')'' = (x' \wedge x'')' = (x' \wedge x)' = 0' = 1$.

Proposition 3.24. *If F is an implicative filter of E, then every filter G of E which contains F is an implicative filter.*

Proof. Let F be an implicative filter. Then, by Theorem 3.18, F is a positive implicative filter. Thus, by Corollary 3.8, G is a positive implicative filter. Suppose $(x \to y) \to y \in G$. By Theorem 3.20, it is enough to prove that $(y \to x) \to x \in G$. For this, let $u = (x \to y) \to y$. Since $u \to ((x \to y) \to y) = 1 \in F$ and F is a positive implicative filter, by Proposition 3.5(iii), $(u \to (x \to y)) \to (u \to y) \in F$. Then, by Proposition 2.8(vii), $(x \to (u \to y)) \to (u \to y) \in F$. Since F is an implicative filter, by Theorem 3.20, $((u \to y) \to x) \to x \in F$. Thus, $((u \to y) \to x) \to x \in G$. By Proposition 2.8(iv) and (v),

$$\begin{aligned}(x \to y) \to y &\leq (((x \to y) \to y) \to y) \to y \\ &= (u \to y) \to y \\ &\leq (y \to x) \to ((u \to y) \to x) \\ &\leq (((u \to y) \to x) \to x) \to ((y \to x) \to x).\end{aligned}$$

By assumption, G is a filter, and so $(((u \to y) \to x) \to x) \to ((y \to x) \to x) \in G$. Since $((u \to y) \to x) \to x \in G$, we have $(y \to x) \to x \in G$. Hence, G is an implicative filter. □

Proposition 3.25. *In any bounded equality algebra E, the following conditions are equivalent*:

(i) $\{1\}$ *is an implicative filter,*
(ii) *every filter of E is an implicative filter,*
(iii) $F(a) = \{x \in E \mid x \geq a\}$ *is an implicative filter, for any $a \in E$,*
(iv) $(x \to y) \to x = x$, *for all $x, y \in E$,*
(v) $x \vee x' = 1$,
(vi) $x' \to x = x$,
(vii) *E is a Boolean algebra.*

Proof. (i) ⇒ (ii) By Proposition 3.24, the proof is clear.

(ii) ⇒ (iii) Since $\{1\}$ is an implicative filter, by Theorem 3.18, $\{1\}$ is a positive implicative filter. Since, for any $a \in E$, $1 \geq a$, we get $1 \in F(a)$. Let $x, x \to y \in F(a)$. Then $a \to x = 1$ and $a \to (x \to y) = 1$. Thus, $a \to y = 1$, and so $y \in F(a)$. Hence, $F(a)$ is a filter. By (ii), $F(a)$ is an implicative filter.

(iii) $\Rightarrow$ (iv) Since $(x \to y) \to x \in F((x \to y) \to x)$, by (iii), $F((x \to y) \to x)$ is an implicative filter. Then, by Proposition 3.16(ii), $x \in F((x \to y) \to x)$, and so $x \geq (x \to y) \to x$. Also, we have $x \leq (x \to y) \to x$. Hence, $x = (x \to y) \to x$.

(iv) $\Rightarrow$ (v) At first, we prove that E is a commutative equality algebra. For this, let $x, y \in E$. Then, by (iv) and Proposition 2.8(v), $(y \to x) \to x = (y \to x) \to ((x \to y) \to x) \geq (x \to y) \to y$. By a similar way, $(x \to y) \to y \geq (y \to x) \to x$. Thus, E is commutative, and so, by Theorem 2.12, E is a lattice such that $x \vee y = (x \to y) \to y$. Hence, by (iv), $x \vee x' = (x' \to x) \to x = ((x \to 0) \to x) \to x = x \to x = 1$.

(v) $\Rightarrow$ (vi) By Proposition 2.10(i) and (v), $x' \to x = (x \vee x') \to x = 1 \to x = x$.

(vi) $\Rightarrow$ (iv) Since $x' \leq x \to y$, by Proposition 2.9(i), $x' \to x \geq (x \to y) \to x$. Then, by (vi), $x \geq (x \to y) \to x$. By Proposition 2.8(iii), $x \leq (x \to y) \to x$, and so $x = (x \to y) \to x$.

(iv)$\Rightarrow$ (vii) If (iv) holds, then E is a bounded commutative equality algebra, and so the condition (v) holds. By Theorem 2.14, every bounded commutative equality algebra can be embedded into an MV-algebra. Thus, the structure $(E, \vee, \wedge, 0, 1)$ is a bounded distributive lattice [2]. Then, by (v) and Lemma 3.23, E is complemention. Therefore, E is a Boolean algebra.

(vii) $\Rightarrow$ (v) The proof is clear.

(v) $\Rightarrow$ (i) Assume that (v) holds, then the condition (vi) holds. Thus, by Proposition 3.17, $\{1\}$ is an implicative filter. □

Corollary 3.26. *Let F be a filter of bounded equality algebra E. Then F is an implicative filter of E if and only if E/F is a Boolean algebra.*

Proof. ($\Rightarrow$) Let F be an implicative filter of E. Then by Proposition 3.17, $(x' \to x) \to x \in F$, for any $x \in E$. Thus, $([x]' \to_F [x]) \to [x] = 1$, and so $([x]' \to_F [x]) \leq [x]$, for any $[x] \in E/F$. Since $x' \leq 1$, by Proposition 2.9(i), $x = 1 \to x \leq x' \to x$. Thus, $x \to (x' \to x) = 1$, and so $[x] \leq [x]' \to_F [x]$. Hence, $[x]' \to_F [x] = [x]$, for any $[x] \in E/F$. By Proposition 3.25(vi), E/F is a Boolean algebra.

($\Leftarrow$) By Propositions 3.25(vi), (vii) and 3.17, the proof is clear. □

Theorem 3.27. *Let F be a filter of E. Then the following conditions are equivalent*:

(i) *F is maximal and implicative filter,*

(ii) *F is maximal and positive implicative filter,*

(iii) *$x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$, for all $x, y \in E$.*

Proof. (i) $\Rightarrow$ (ii) By Theorem 3.18, the proof is clear.

(ii) $\Rightarrow$ (iii) Suppose F is a positive implicative filter and $x, y \notin F$. By Proposition 3.11, F_y is the least filter containing F and y, that is, $F \subsetneqq F_y \subseteq E$. Since F is a maximal filter, $F_y = E$. Then $x \in F_y$, and so $y \to x \in F$. By a similar way, since $x \notin F$, $x \to y \in F$.

(iii) $\Rightarrow$ (i) Suppose F is not an implicative filter. Then, by Proposition 3.16(ii), there exist $x, y \in E$ such that $(x \to y) \to x \in F$, and $x \notin F$. If $y \in F$, since $y \leq x \to y$ and F is a filter, $x \to y \in F$. Thus, $x \in F$, which is a contradiction. If $y \notin F$, then by (iii), $x \to y \in F$, since F is a filter and $(x \to y) \to x \in F$, $x \in F$, which is a contradiction. Hence, F is an implicative filter. Now, we prove that F is a maximal filter. Let G be a filter of E such that $F \subsetneq G \subseteq E$ and $a \in G \setminus F$. Since F_a is the least filter containing F and a, we have $F \subseteq F_a \subseteq G \subseteq E$. Let $u \in E$. If $u \in F$, then $u \in F_a$. If $u \notin F$, since $a \notin F$, then $a \to u \in F$. By Proposition 3.11, $u \in F_a$. Thus, $F_a = E$, and so $G = E$. Hence, F is a maximal filter. $\square$

Corollary 3.28. *Let F be a maximal filter of E. Then F is an implicative filter if and only if F is a positive implicative filter.*

4 Fantastic filters in equality algebra

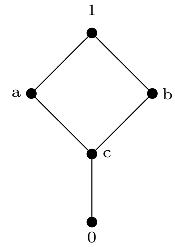
In this section, we introduce the notion of fantastic filters of equality algebras and investigate some properties of them.

Definition 4.1. A non-empty subset F of E is called a *fantastic filter* if

(i) $1 \in F$,

(ii) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$, for all $x, y, z \in E$.

Example 4.2. Let $(E = \{0, a, b, c, 1\}, \leq)$ be a lattice with the following diagram. Define the operations $\sim$ and $\rightarrow$ on E as follows,



$\sim$	0	c	a	b	1
0	1	0	0	0	0
c	0	1	b	a	c
a	0	b	1	c	a
b	0	a	c	1	b
1	0	c	a	b	1

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then, by routine calculations, we can see that $(E, \wedge, \sim, 1)$ is an equality algebra and $\{1, a, b, c\}$ is a fantastic filter.

Lemma 4.3. *Any fantastic filter of E is a filter.*

Proof. Let $z, z \rightarrow x \in F$. Since $z \rightarrow (1 \rightarrow x) = z \rightarrow x \in F$ and F is a fantastic filter, we have $((x \rightarrow 1) \rightarrow 1) \rightarrow x = x \in F$. Hence, F is a filter. □

Example 4.4. Let E be an equality algebra as Example 4.2. By routine calculations, we can see that $F = \{1, a\}$ is a filter which is not a fantastic filter. Because, $0 \rightarrow b = 1 \in F$, but $((b \rightarrow 0) \rightarrow 0) \rightarrow b = b \notin F$.

Proposition 4.5. *Let F be a filter of E. Then the following conditions are equivalent*:

(i) *F is a fantastic filter of E,*

(ii) *$y \rightarrow x \in F$ implies $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$, for all $x, y \in E$,*

(iii) *if E is a lattice, then $((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) \in F$, for all $x, y \in E$.*

Proof. (i) $\Rightarrow$ (ii) Suppose F is a fantastic filter and $y \rightarrow x \in F$. Let $z = 1$. Since $1 \rightarrow (y \rightarrow x) = y \rightarrow x \in F$ and $1 \in F$, $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$.

(ii) $\Rightarrow$ (i) Since F is a filter, if $z \rightarrow (y \rightarrow x) \in F$ and $z \in F$, then $y \rightarrow x \in F$. Thus by (ii), $((x \rightarrow y) \rightarrow y\ ) \rightarrow x \in F$.

(ii) $\Rightarrow$ (iii) Let E be a lattice. By Proposition 2.10(i), $(x \vee y) \to y = x \to y$ and $y \to (x \vee y) = 1 \in F$, for any $x, y \in E$. Then, by (ii), $(((x \vee y) \to y) \to y) \to (x \vee y) \in F$, and so $((x \to y) \to y) \to (x \vee y) \in F$. Moreover, by Proposition 2.8(iii) and (iv), $x \leq (y \to x) \to x$ and $y \leq (y \to x) \to x$. Thus, $x \vee y \leq (y \to x) \to x$, and so $(x \vee y) \to ((y \to x) \to x) = 1 \in F$. Then, by Lemma 3.6(ii), $((x \to y) \to y) \to ((y \to x) \to x) \in F$.

(iii) $\Rightarrow$ (ii) Let $x, y \in E$ and $y \to x \in F$. Since $(y \to x) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((y \to x) \to x) \in F$ and F is a filter, we have $((x \to y) \to y) \to x \in F$. $\square$

Corollary 4.6. *In a commutative equality algebra, any filter is a fantastic filter.*

Proof. By Theorem 2.12 and Proposition 4.5(iii), the proof is clear. $\square$

Proposition 4.7. *Suppose F and G are two filters of E and $F \subseteq G$. If F is a fantastic filter, then so is G.*

Proof. Let $y \to x \in G$, for any $x, y \in E$. By Proposition 2.8(vii), $y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1 \in F$. Since F is a fantastic filter, $((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x) \in F \subseteq G$. Thus, by Proposition 2.8(vii), $(y \to x) \to (((((y \to x) \to x) \to y) \to y) \to x) \in G$. Since G is a filter and $y \to x \in G$, $((((y \to x) \to x) \to y) \to y) \to x \in G$. By Proposition 2.8(v) and (vii), $((((((y \to x) \to x) \to y) \to y) \to x)) \to (((x \to y) \to y) \to x) \geq x \to ((y \to x) \to x) = (y \to x) \to (x \to x) = 1 \in G$. Since G is a filter, $((((((y \to x) \to x) \to y) \to y) \to x)) \to (((x \to y) \to y) \to x) \in G$, and so $((x \to y) \to y) \to x \in G$. Hence, G is a fantastic filter. $\square$

Corollary 4.8. *Every filter of E is a fantastic filter if and only if $\{1\}$ is a fantastic filter.*

Proposition 4.9. *An equality algebra E is commutative if and only if $\{1\}$ is a fantastic filter.*

Proof. ($\Rightarrow$) Suppose E is a commutative equality algebra. Since $\{1\}$ is a filter, by Corollary 4.6, it is a fantastic filter.

($\Leftarrow$) Let $\alpha = (y \to x) \to x$, for $x, y \in E$. Then $y \to \alpha = y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1$. Since $\{1\}$ is a fantastic filter, $((\alpha \to y) \to y) \to \alpha \in \{1\}$. Since $x \leq \alpha$, by Proposition 2.9(i),

$((\alpha \to y) \to y) \to \alpha \leq ((x \to y) \to y) \to \alpha$. Then $((x \to y) \to y) \to \alpha = 1$, and so $(x \to y) \to y \leq (y \to x) \to x$. By a similar way, $(y \to x) \to x \leq (x \to y) \to y$. Hence, E is commutative. □

Proposition 4.10. *Let F be a filter of E. Then F is a fantastic filter if and only if every filter of E/F is a fantastic filter.*

Proof. ($\Rightarrow$) Suppose F is a fantastic filter of E and $x, y \in E$ such that $[x] \to_F [y] = [1]$. Then, $x \to y \in F$. Thus, $((y \to x) \to x) \to y \in F$, and so $(([y] \to_F [x]) \to_F [x]) \to_F [y] = [((y \to x) \to x) \to y] = [1]$, which proves that $\{[1]\}$ is a fantastic filter of E/F. By Corollary 4.8, every filter of E/F is a fantastic filter.

($\Leftarrow$) Let $x, y \in E$ such that $x \to y \in F$. Then $[x] \to_F [y] = [1]$. Since $\{[1]\}$ is a fantastic filter of E/F, we have $[((y \to x) \to x) \to y] = [1]$, and so $((y \to x) \to x) \to y \in F$. Thus, F is a fantastic filter of E. □

Corollary 4.11. *F is a fantastic filter of E if and only if E/F is a commutative equality algebra.*

Proof. Let F be a filter of E. By Proposition 4.10, Corollary 4.8, and Proposition 4.9, F is a fantastic filter of E if and only if every filter of E/F is a fantastic filter if and only if $\{[1]\}$ is a fantastic filter if and only if E/F is a commutative equality algebra. □

Theorem 4.12. *Any implicative filter of E is a fantastic filter.*

Proof. Let F be an implicative filter of E and $y \to x \in F$. Since $x \leq ((x \to y) \to y) \to x$, by Proposition 2.9(i), $(((x \to y) \to y) \to x) \to y \leq x \to y$. Then, by Proposition 2.9(i),

$$\begin{aligned}
&((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \\
&\geq (x \to y) \to (((x \to y) \to y) \to x), \text{ by Proposition 2.8(vii)} \\
&= ((x \to y) \to y) \to ((x \to y) \to x), \text{ by Proposition 2.9(iii)} \\
&\geq y \to x
\end{aligned}$$

By Lemma 4.3, F is a filter, and so, $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \in F$ and, by Proposition 3.16(ii), $((x \to y) \to y) \to x \in F$. □

Example 4.13. Let $(E = \{0, a, b, 1\}, \leq)$ be a chain. Define the operations $\sim$ and $\to$ on E by

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

By routine calculations, we can see that $(E, \sim, \wedge, 0, 1)$ is an equality algebra and $F = \{1, b\}$ is a fantastic filter, which is not an implicative filter. Because, $(a \to 0) \to a = 1 \in F$, but $a \notin F$.

Notation. In Example 3.2, $F = \{1, b\}$ is a positive implicative filter, but it is not a fantastic filter. Because $0 \to a = 1 \in F$, but $((a \to 0) \to 0) \to a = a \notin F$. Moreover, in Example 3.4, $G = \{1, c\}$ is a fantastic filter, but it is not a positive implicative filter. Because $a \to (a \to b) = 1 \in G$, but $a \to b = a \notin G$. This shows that positive implicative and fantastic filters do not coincide, in general.

Theorem 4.14. *F is an implicative filter of E if and only if F is a positive implicative filter and fantastic filter.*

Proof. $(\Rightarrow)$ By Theorems 3.18 and 4.12, the proof is clear.

$(\Leftarrow)$ Let $x, y \in E$ and $(x \to y) \to x \in F$. By Proposition 2.8(v), $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$. Then, by Lemma 3.3, F is a filter, $(x \to y) \to ((x \to y) \to y) \in F$. Since F is a positive implicative filter, by Proposition 3.5(ii), $(x \to y) \to y \in F$. Moreover, by Propositions 2.8(iii) and 2.9(i), $(x \to y) \to x \leq y \to x$. Thus, $y \to x \in F$. Since F is a fantastic filter, $((x \to y) \to y) \to x \in F$. Since $(x \to y) \to y \in F$ and F is a filter, $x \in F$. Hence, by Proposition 3.16(ii), F is an implicative filter of E. $\square$

5 Boolean and prime filters in equality algebras

In this section, we introduce the notions of Boolean and prime filters in equality algebras and investigate some of their properties.

Definition 5.1. Let E be a bounded lattice equality algebra. A filter F of E is called a *Boolean filter* if, for all $x \in E$, $x \vee x' \in F$.

Example 5.2. Let E be an equality algebra as in Example 4.2. By routine calculations, we can see that $F = \{1, a, b, c\}$ is a Boolean filter of E.

Theorem 5.3. *Suppose E is a bounded lattice equality algebra and F is a filter of E. Then F is a Boolean filter if and only if F is an implicative filter.*

Proof. ($\Rightarrow$) Let F be a Boolean filter. Then for any $x \in E$, $x \vee x' \in F$. If $x' \to x \in F$, then, by Proposition 2.10(ii), $(x \vee x') \to x = (x \to x) \wedge (x' \to x) = x' \to x \in F$. Since F is a filter, $x \in F$.

($\Leftarrow$) Suppose F is an implicative filter. Since $x' \wedge x'' \leq x' \leq x' \vee x$, by Proposition 2.11(ii), we have $(x \vee x')' \to (x \vee x') = (x' \wedge x'') \to (x \vee x') = 1 \in F$. Hence, by Proposition 3.16(iii), $x \vee x' \in F$. □

Corollary 5.4. *Suppose E is a bounded equality algebra. Then E is a Boolean algebra if and only if $\{1\}$ is a Boolean filter.*

Proof. From Proposition 3.25 and Theorem 5.3, the proof is clear. □

Corollary 5.5. *Each Boolean filter of a bounded lattice equality algebra is a positive implicative and a fantastic filter.*

Proof. By Theorems 3.18, 4.12, and 5.3, the proof is clear. □

The following example shows that the converse of Corollary 5.5 may not be true, in general.

Example 5.6. (i) In Example 3.4, $F = \{1, c\}$ is a fantastic filter, but it is not a Boolean filter. Because, $a \vee a' = a \notin F$.

(ii) In Example 3.2, $G = \{1, b\}$ is a positive implicative filter, but it is not a Boolean filter. Because, $a \vee a' = a \notin G$.

Corollary 5.7. *In any bounded commutative equality algebra, implicative, positive implicative, and Boolean filters coincide.*

Proof. By Corollary 4.6, Theorems 4.14, 2.12, and 5.3, the proof is clear. □

Definition 5.8. A proper filter F of E is called a *prime filter* if $x \to y \in F$ or $y \to x \in F$, for all $x, y \in E$.

Example 5.9. In Example 4.2, $\{1, a\}$, $\{1, b\}$, and $\{1, a, b, c\}$ are prime filters.

Theorem 5.10. *Let F be a filter of lattice equality algebra E. Then the following statements are equivalent:*

(i) *F is a maximal and Boolean filter,*

(ii) *F is a maximal and positive implicative filter,*

(iii) *$x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$, for all $x, y \in E$,*

(iv) *F is a prime and Boolean filter,*

(v) *F is a proper filter such that $x \in F$ or $x' \in F$, for every $x \in E$.*

Proof. By Theorems 5.3 and 3.27, the proof of (i)⇒ (ii), (ii)⇒(iii), and (iii)⇒ (i) are clear.

(i)⇒(iv) Suppose F is not a prime filter. Then there exist $x, y \in E$ such that $x \to y \notin F$ and $y \to x \notin F$. Since $x \leq y \to x$ and F is a filter, $x \notin F$. By Proposition 3.11, F_x is the least filter containing F and x. Also, by assumption, F is a maximal filter, and so $F_x = E$. Thus, $y \in F_x$, and so $x \to y \in F$, which is a contradiction.

(iv)⇒(v) Let F be a prime and Boolean filter. Then, for any $x \in E$, $x \to x' \in F$ or $x' \to x \in F$. If $x \to x' \in F$, then, by Proposition 2.10(ii), $(x \vee x') \to x' = (x \to x') \wedge (x' \to x') = x \to x' \in F$. Since F is a Boolean filter, $x' \in F$. By a similar way, if $x' \to x \in F$, then $x \in F$.

(v)⇒(i) Let F be a proper filter such that satisfies (v). If $x \in F$, then $x \vee x' \in F$. If $x \notin F$, then by (v), $x' \in F$, and so $x \vee x' \in F$. Hence, F is a Boolean filter. Now, we prove that F is a maximal filter. Let G be a proper filter of E such that $F \subseteq G \subsetneqq E$. If $x \in G \setminus F$, then $x' \in F$, and so $x' \in G$. Hence, $0 \in G$, which is a contradiction. □

Theorem 5.11. *Let F be a proper filter of prelinear equality algebra E. Then the following statements are equivalent:*

(i) *F is a prime filter,*

(ii) *for each $x, y \in E$, if $x \vee y \in F$, then $x \in F$ or $y \in F$,*

(iii) *E/F is a chain or equivalently $\leq_F$ is totally ordered.*

Proof. *(i) ⇒ (ii)* Suppose F is a prime filter and $x \vee y \in F$. Since E is prelinearly, $(x \to y) \vee (y \to x) = 1 \in F$. Since F is prime, $x \to y \in F$, by Proposition 2.10(i), $(x \vee y) \to y \in F$. Thus, $y \in F$.

$(ii) \Rightarrow (iii)$ Since E is prelinear, by (ii), $x \rightarrow y \in F$ or $y \rightarrow x \in F$. Thus, F is a prime, and so $[x] \leq_F [y]$ or $[y] \leq_F [x]$. Hence, E/F is a chain.

$(iii) \Rightarrow (ii)$ The proof is clear. $\square$

Corollary 5.12. *Let F be a proper filter of prelinear equality algebra E. Then F is a prime filter if and only if E/F is a chain.*

6 Conclusion

It is well-known that using filters with special properties plays an important role in investigating the structure of a logical system. From a logical point of view, the sets of provable formulas can be described by fuzzy filters of those algebraic semantics. Moreover, the properties of filters have a strong influence on the structure properties of algebras. In this study, we proposed the concepts of (positive) implicative, fantastic, and Boolean filters in equality algebras and investigated several of their properties. We established the relations between these filters and quotient structures which are constructed via them.

There are still some open problems. In BL-algebras, the quotient structures induced by positive implicative filters are Gödel algebras. In [10], the authors proved that the quotient structures induced by positive implicative filters in residuated EQ-algebras are idempotent residuated EQ-algebras. What is the quotient structures induced by a positive implicative filter in an equality algebra? Moreover, In [9], it is proved that an MTL-algebra has states if and only if it has a fantastic filter. What is the relation between fantastic filters and states on equality algebras? These could be a topic of further research.

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