Categories and
General Algebraic Structures
with Applications
Volume 11, Special Issue Dedicated to Professor George A. Grätzer,
July 2019, 1-9.

Celebrating Professor George A. Grätzer

Gábor Czédli

Introduction It is a great honor for me to write a few introductory words to the present volume of CGASA **dedicated to Professor George A. Grätzer**. The occasion for this dedication is that we are celebrating two anniversaries in 2018 related to him. Namely,

(A1) it was 55 years ago that the Grätzer–Schmidt Theorem was published, and

(A2) it was 40 years ago that G. Grätzer's General Lattice Theory [17], which immediately became the Book in lattice theory for decades, appeared.

George (in Hungarian, György) Grätzer was born in Budapest, the capital of Hungary, in 1936. He and his close friend, E. Tamás Schmidt made their first research steps in mathematics while being undergraduate students at Eötvös University, Budapest. As the list of their publications given in [3] shows, they published five research papers in 1957, when they both were

Keywords: George Grätzer, lattice theory, Grätzer–Schmidt Theorem.
Mathematics Subject Classification[2010]: 01A70.
ISSN: Print 2345-5853, Online 2345-5861.

only 21, and six in the next year. Their longstanding collaboration has resulted in 61 nice joint papers. On the occasion of their seventieth birthdays, B. Beeton, L. Fuchs, and I devoted a paper to their life and work; see [2]. Also, I wrote a twenty page long survey on their contributions to lattice theory; see [3]. Since space considerations do not allow to repeat all what are included in [2] and [3], I will mainly focus on (A(A1)), (A(A2)), and Grätzer's activity in the last decade, though the present note is not entirely disjoint from [2] and [3]. Following another guiding principle, this note is intended to be readable without assuming too much from lattice theory.

I have to admit that even the union of (A(A1)), (A(A2)), the present paper, and a forthcoming interview with him is far from being complete in any sense. Professor Grätzer is one of the most prolific and most influential expert among all lattice theorists who have ever lived; personally, I rank him among the top three. He was the thesis advisor of fourteen successful Ph.D. students; see [2] for their list. According to MathSciNet, he has more than 270 publications, whence it would be hopeless to give a satisfactory account of his achievements in the union of the above-mentioned four papers; such an ambitious plan would require several authors and several books.

The Grätzer–Schmidt Theorem and thereafter As we have already mentioned in (A(A1)), G. Grätzer and E.T. Schmidt published a very deep theorem 55 years ago; see [36]. This theorem, which is nowadays regularly included in any course on lattices or universal algebra, is the *Grätzer–Schmidt Theorem.* Solving a famous problem raised by G. Birkhoff, this theorem characterizes the congruence lattices $\mathrm{Con}(A)$ of algebraic structures A as complete lattices in which every element is the join of compact elements.

The Grätzer–Schmidt Theorem is the start of many deep and interesting results. This is clear by the following table even if it serves only as an approximation indicating the trends.

Search command	Matches
"Author=(Gratzer, G*) AND Title=(congruence lattice)"	78
"Title=(congruence lattice)"	712

Table 1: Search in MathSciNet

The research motivated by the Grätzer–Schmidt Theorem culminated in Wehrung [46] by proving the existence of a *distributive* algebraic lattice that cannot be represented as the congruence lattice of a *lattice*. Such a non-representable lattice is necessarily infinite. Wehrung's theorem shows that one can expect nice results mainly on the characterization of congruence lattices of *finite* lattices. There are two natural targets in this directions: to represent a finite distributive lattice D as the congruence lattice of a finite lattice with *nice properties* and to represent two or more related finite distributive lattices *simultaneously*. These two targets are not disjoint. Many results of these kinds are reported in Grätzer [28]; mostly his (joint) results.

In spite of its importance and deepness, the Grätzer–Schmidt Theorem is not at all the deepest common result of its authors. I mention only one example of a much deeper result, by which I am particularly impressed. In [40], they proved that

> every complete lattice L is representable as the lattice $\mathrm{Com}(D)$ of complete congruences of a suitable complete *distributive* lattice D. (0.1)

Note that whenever L is not distributive, then D is necessarily infinite. In order to enlighten the difficulty of finding an appropriate construction and using it to prove (0.1), we conclude this paragraph with some historical comments. By Funayama and Nakayama [15], it has been known since 1942 that $\mathrm{Con}(K)$ for a lattice K is always distributive, but it took four and a half decades to recognize that this is not so for $\mathrm{Com}(K)$; see Reuter and Wille [44]. Soon afterwards, Teo [45] represented every finite lattice L in the form $\mathrm{Com}(K)$, without requiring the distributivity of K; the date of [45] shows some publication delay. In 1988, a year after that Reuter and Wille [44] appeared, Gätzer [18] and [19] announced and outlined that every complete lattice L is of the form $\mathrm{Com}(K)$ for a suitable complete lattice K, but the full proof of this fact became available only in 1991; see Grätzer and Lakser [34]. In the same year, Freese, Grätzer, and Schmidt [14] proved that K above can be chosen to be modular, but there still remained a serious obstacle on the way towards (0.1). Namely, we have already mentioned that D in (0.1) has to be infinite in general but no infinite complete distributive lattice D even with the very modest requirement $|\mathrm{Com}(D)| = 2$ was known that time. It took two years to find such a D in Grätzer and Schmidt [37] and [38], and then two additional years were needed to prove (0.1) in Grätzer and Schmidt [40] in 1995. In the same year, they also showed in [39] that the 1993 papers [37] and [38] giving an infinite complete distributive lattice D with $|\mathrm{Com}(D)| = 2$ were not a waist of time since any construction proving (0.1) finds such a D in some sense.

The story of (0.1) in the paragraph above is connected with Grätzer's taste about lattice theory. His papers and books witness that, with my temporary adjectives, he prefers the "nice and elegant" classical lattice theory to its applications. Of course, applications are as important as my adjectives expressing beauty but, to tell the truth, when I first met Reuter and Wille [44] and Teo [45] mentioned in the previous paragraph, I felt a little disappointment. To prove their results, [44] and [45] used the most applicable part of lattice theory, Wille's *Formal Concept Analysis.* I had the feeling that the beauty of Hasse diagrams and purely lattice theoretical methods are not as powerful as Formal Concept Analysis. By 1995, my disappointment was over, because all papers mentioned after Teo [45] in the previous paragraph belong to the nice and elegant classical lattice theory.

In the same year when the Grätzer–Schmidt Theorem [36] appeared, George Grätzer moved to the United States and became a professor of mathematics at Pennsylvania State University. Three years later, he became a professor at the University of Manitoba. Now he is an emeritus professor of the same university, but he lives in Toronto. He is a Canadian citizen, he is married with two sons and five grandchildren.

In 1971, he founded Algebra Universalis, which prompt became and still is the main journal for the community of lattice theorists and universal algebraists. Furthermore, he served as a founding editor-in-chief of Algebra Universalis for forty five years, 1971–2016. It would be hard to overestimate the great importance of this service for us.

General Lattice Theory The appearance of Grätzer's Lattice Theory [17] forty years ago was, again, an important event in the history of lattice theory. This well-written monograph with almost 900 exercises, 193 open problems, and a rich bibliography containing over 750 items became the Reference Book for lattice theorists immediately, much before the era of Internet. Generations of lattice theorists have grown on this book. I fully agree to M. Stern, who wrote in Mathematical Reviews, MR509213, that "even a first scanning of this rich book shows what lattice theory is like, and gives the reader an impression of the beauty of this comparatively young branch of abstract algebra."

The rapid development of lattice theory, which was partly due to the General Lattice Theory [17], made it necessary that its second edition in 1998 was augmented with several appendices written by Grätzer himself and several invited authors; see [20] in the References section of the present paper. While [17] in 1978 consisted of “only” xiii+381 pages, its extended 1998 version [20] was xx+663 page long. This fact and further developments in lattice theory made Grätzer to realize that a single book would not be sufficient for the next edition. Hence, he alone wrote the "foundation book” [24] in 2011, which were followed by two “special topics an applications" volumes, [41] from 2014 and [42] from 2016. Each of these two volumes was jointly edited by G. Grätzer and F. Wehrung and consists of ten chapters, written by several authors. Grätzer himself wrote four chapters of [41], one of them with a coauthor. Forty years after that the General Lattice Theory first appeared, its descendant, the three-volume set consisting of [24], [41], and [42], is still the most important book in lattice theory.

George Grätzer’s achievements in the last decade At the beginning of my scientific career, I learned a lot from what we are celebrating now, see (A(A1)) and (A(A2)), and also from Professor Grätzer’s journal, Algebra Universalis. Later, George and I became friends and, in the last decade, coauthors; see [8–11].

In the last decade, George Grätzer worked a lot as the author or one of the editors on new and/or extended editions of many of his mathematical books, [21, 24, 28, 41, 42], and even on some other books, [23, 25, 27, 29, 30]. It is worth mentioning that [29] on LaTeX is his ninth book on mathematical typesetting; see the bibliographic section of [3] for the list of the previous eight. He is well known for most mathematicians as an author of excellent books on LaTeX. Note that the “puzzle book” [25] as well as [22] and [23] are translations and a new edition of his first book [16], appeared in 1959, while [30] on chess endgames is based on his very first publications in 1954–1956.

Also in the last decade, in addition to his four book chapters included in [41], he published 28 research papers on his new mathematical results, 26 of them belonging to pure lattice theory. These 28 papers came to existence after [3], where his publications up to 2008 are listed. Many of these 28 recent papers are devoted to planar semimodular lattices and their congruences; most of his results of this kind are surveyed in our book chapter [10]. Besides that this topic has already lead to many interesting results in lattice theory, it has influenced further research even outside lattice theory. For example, according to page 62 of Czédli [4], the progress due to Grätzer and Knapp [31–33] (and some other papers based on these three but not listed here) "provides the background of" [4]. Since [4] was continued in Adaricheva and Bolat [1], Czédli [5–7], Czédli and Kincses [12], Czédli and Stachó [13], and Kincses [43], we have that Grätzer and Knapp's above-mentioned papers on planar semimodular lattices have already lead, directly or indirectly, to eight papers belonging to combinatorial geometry or geometry.

Another sub-series of his recent 28 papers started with [26], which is a deep paper proving that every bounded ordered set, also known as a bounded poset, can be represented as the ordered set $\mathrm{Princ}(L)$ of principal congruences of a lattice L of length at most five. One of his most recent results, Grätzer and Lakser [35], aims at simultaneous characterizations of a finite bounded ordered set and a finite distributive lattice, subject to some conditions, as $\mathrm{Princ}(L)$ and $\mathrm{Con}(L)$, respectively; this is again a new exciting topic due to him.

Summary As my short writing shows, George Grätzer is a living legend of lattice theory. He is still active, he still proves a lot of interesting theorems and motivates further research in this theory.

I wish to Professor Grätzer, my friend George, and also to all of us that he continue proving a lot of nice theorems and raising a lot of interesting problems for many years to come.

References

[1] Adaricheva, K. and Bolat, M., *Representation of convex geometries by circles on the plane*, https://arxiv.org/abs/1609.00092 .

[2] Beeton, B., Czédli, G., and Fuchs, L., *Reminiscences about George Grätzer and E. Tamás Schmidt*, Algebra Universalis **59** (2008), 3–10.

[3] Czédli, G., *The mathematics of G. Grätzer and E.T. Schmidt*, Algebra Universalis **59** (2008), 11–30.

[4] Czédli, G., *Finite convex geometries of circles*, Discrete Mathematics **330** (2014), 61–75.

[5] Czédli, G., *Characterizing circles by a convex combinatorial property*, Acta Sci. Math. (Szeged) **83** (2017), 683–701.

[6] Czédli, G., *An easy way to a theorem of Kira Adaricheva and Madina Bolat on convexity and circles*, Acta Sci. Math. (Szeged) **83** (2017), 703–712.

[7] Czédli, G., *Circles and crossing planar compact convex sets*, Acta Sci. Math. (submitted); https://arxiv.org/pdf/1802.06457 .

[8] Czédli, G. and Grätzer, G., *Lattice tolerances and congruences*, Algebra Universalis **66** (2011), 5–6.

[9] Czédli, G. and Grätzer, G., *Notes on planar semimodular lattices. VII. Resections of planar semimodular lattices*, Order **30** (2013), 847–858.

[10] Czédli, G. and Grätzer, G., *Planar semimodular lattices: structure and diagrams*, in "Lattice theory: special topics and applications. Vol. 1", 91–130, Birkhäuser/Springer, 2014.

[11] Czédli, G., Grätzer, G., and Lakser, H., *Congruence structure of planar semimodular lattices: the General Swing Lemma*, Algebra Universalis **79** (2018), no. 2, 79:40. DOI 10.1007/s00012-018-0483-2.

[12] Czédli, G. and Kincses, J., *Representing convex geometries by almost-circles*, Acta Sci. Math. (Szeged) **83** (2017), 393–414.

[13] Czédli, G. and Stachó, L.L., *A note and a short survey on supporting lines of compact convex sets in the plane*, Acta Univ. M. Belii Ser. Math. **24** (2016), 3–14.

[14] Freese, R., Grätzer, G., and Schmidt, E.T., *On complete congruence lattices of complete modular lattices*, Internat. J. Algebra Comput. **1** (1991), 147–160.

[15] Funayama, N. and Nakayama, T., *On the distributivity of a lattice of lattice-congruences*, Proc. Imp. Acad. Tokyo **18** (1942), 553–554.

[16] Grätzer, G., "Elmesport egy esztendőre" Gondolat Kiadó, Budapest, 1959 (Hungarian; see [25] for the English edition).

[17] Grätzer, G., "General Lattice Theory", Academic Press, 1978.

[18] Grätzer, G., *On the automorphism group and the complete congruence lattice of a complete lattice*, Abstract 88T-05-215, Abstracts Amer. Math. Soc. **9** (1988), 416.

[19] Grätzer, G., *The complete congruence lattice of a complete lattice*, in Almeida, J., Bordalo, G., Dwinger, Ph. (eds) "Lattices, semigroups, and universal algebra", pp. 81–87. (Proc. Conf. Lisbon, 1988) Plenum Press, New York (1990)

[20] Grätzer, G., "General lattice theory" Second edition, with new appendices by the author together with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung and R. Wille, Birkhäuser Verlag, Basel, 1998. xx+663 pp.

[21] Grätzer, G., "Universal algebra", With appendices by Grätzer, Bjarni Jónsson, Walter Taylor, Robert W. Quackenbush, Günter H. Wenzel, and Grätzer and W. A. Lampe. Revised reprint of the 1979 second edition. Springer, New York, 2008.

[22] Grätzer, G., "Elmesport egy esztendőre", 2nd edition, Nyitott Könyvműhely, Budapest, 2008 (Hungarian; see [25] for the English edition), 238 pp., ISBN: 978-963-9725-36-2.

[23] Grätzer, G., "Denksport für ein Jahr: 140 mathematische Rätsel aus dem Alltag", Spektrum Akademischer Verlag, 2010 (German; see [25] for the English edition).

[24] Grätzer, G., "Lattice theory: Foundation", Birkhäuser/Springer, 2011.

[25] Grätzer, G., "Train Your Brain: A Year's Worth of Puzzles", A K Peters/CRC Press, 2011, ISBN-10: 156881710X, ISBN-13: 978-1568817101.

[26] Grätzer, G., *The order of principal congruences of a bounded lattice*, Algebra Universalis **70** (2013), 95–105.

[27] Grätzer, G., "Practical LaTeX", Springer-Verlag, New York, 2014. ISBN: 978-3-319-06424-6

[28] Grätzer, G., "The congruences of a finite lattice. A 'proof-by-picture' approach", second edition. Birkhäuser/Springer, 2016.

[29] Grätzer, G., "More Math Into LaTeX", fifth edition, Springer, 2016

[30] Grätzer, G., "Selected Chess Compositions", http://server.math.umanitoba.ca/~gratzer/images/pdfs/chess.pdf or http://www.arves.org/arves/index.php/en/endgamestudies/studies-by-composer/984-gratzer-george-1936 .

[31] Grätzer, G. and Knapp, E., *Notes on planar semimodular lattices. I. Construction*, Acta Sci. Math. (Szeged) **73** (2007), 445–462.

[32] Grätzer, G. and Knapp, E., *Notes on planar semimodular lattices.III. Congruences of rectangular lattices*, Acta Sci. Math. (Szeged) **75** (2009), 29–48.

[33] Grätzer, G. and Knapp, E., *Notes on planar semimodular lattices. IV. The size of a minimal congruence lattice representation with rectangular lattices*, Acta Sci. Math. (Szeged) **76** (2010), 3–26.

[34] Grätzer, G. and Lakser, H., *On complete congruence lattices of complete lattices*, Trans. Amer. Math. Soc. **327** (1991), 385–405.

[35] Grätzer, G. and Lakser, H., "Minimal representations of a finite distributive lattice by principal congruences of a lattice", Acta Sci. Math. (Szeged), to appear.

[36] Grätzer, G. and Schmidt, E.T., *Characterizations of congruence lattices of abstract algebras*, Acta Sci. Math. (Szeged) **24** (1963), 34–59.

[37] Grätzer, G. and Schmidt, E.T., *'Complete-simple' distributive lattices*, Proc. Amer. Math. Soc. **119** (1993), 63–69.

[38] Grätzer, G. and Schmidt, E.T., *Another construction of complete-simple distributive lattices*, Acta Sci. Math. (Szeged) **58** (1993), 115–126.

[39] Grätzer, G. and Schmidt, E.T., *Do we need complete-simple distributive lattices?*, Algebra Universalis 33 (1995), 140–141.

[40] Grätzer, G. and Schmidt, E.T., *Complete congruence lattices of complete distributive lattices*, J. Algebra **171** (1995), 204–229.

[41] Grätzer, G. and Wehrung, F.: "Lattice theory: special topics and applications. Vol. 1", edited by George Grätzer and Friedrich Wehrung, Birkhäuser/Springer, 2014.

[42] Grätzer, G. and Wehrung, F., "Lattice theory: special topics and applications. Vol. 2", edited by George Grätzer and Friedrich Wehrung, Birkhäuser/Springer, 2016.

[43] Kincses, J., *On the representation of finite convex geometries with convex sets*, Acta Sci.Math. (Szeged) **83** (2017), 301–312.

[44] Reuter, K. and Wille, R., *Complete congruence relations of concept lattices*, Acta Sci. Math. (Szeged) **51** (1987), 319–327.

[45] Teo, S.-K., *Every finite lattice is a complete congruence lattice*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **33** (1990), 177–182.

[46] F. Wehrung, *A solution of Dilworth's congruence lattice problem*, Adv. Math. **216** (2007), 610–625.

Gábor Czédli, *Bolyai Institute, University of Szeged, Szeged, Aradi vértanúacute;k tere 1, H6720 Hungary*

Email: czedli@math.u-szeged.hu